fair division of the indivisibles

Rohit Vaish

Slides available at

https://rohitvaish.in

Fair Division

Fair Division

Fair Division

Fair Division

Fair Division

Fair Division

Fair Division

Divisible

Fair Division

Divisible

Indivisible

Fair Division

Divisible

Indivisible

The Model

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A B C D E

The Model

	A	B	C	D	E
	4	1	2	2	2
	1	0	5	1	1
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Additive valuations

Envy-Freeness [Gamow and Stern, 1958; Foley, 1967]

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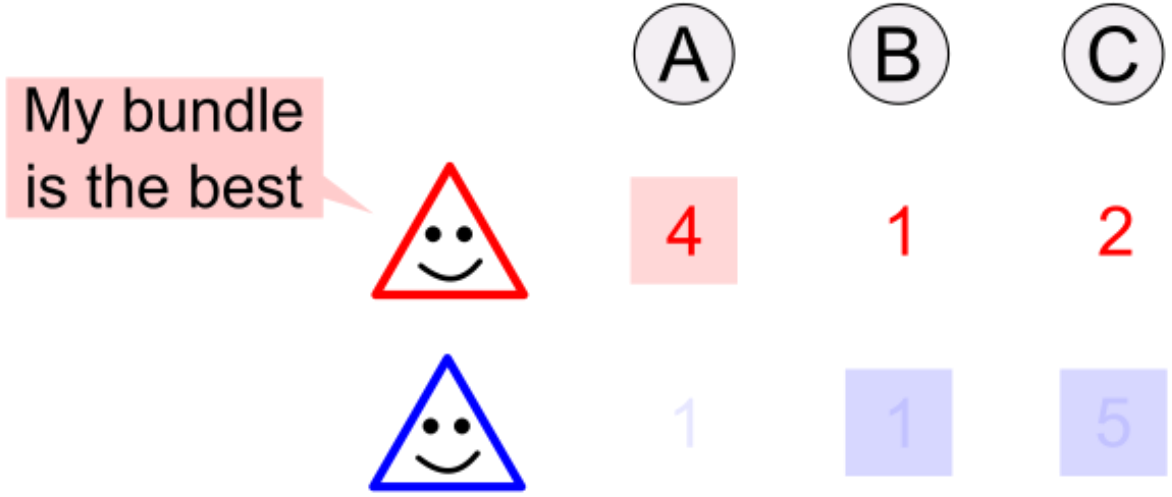

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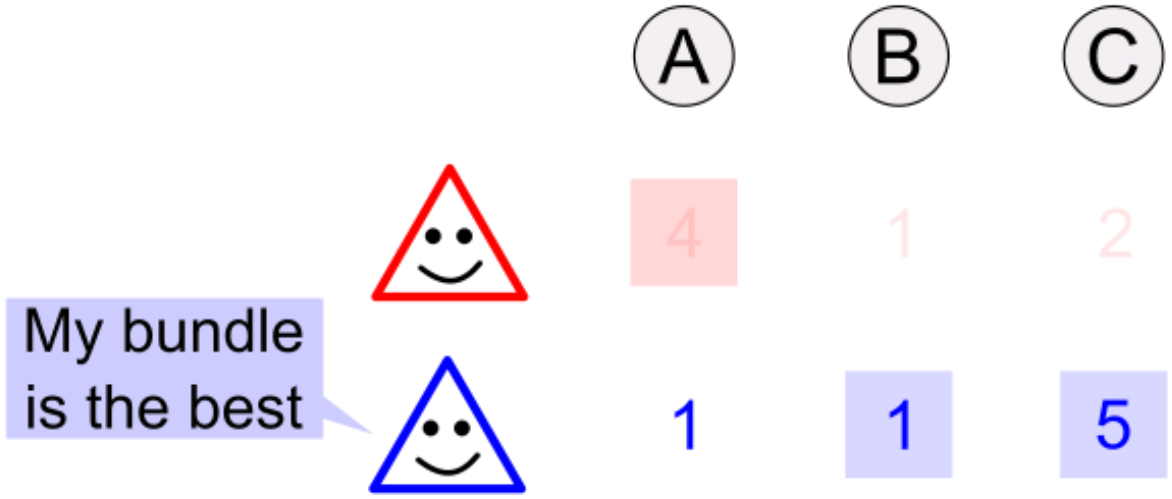

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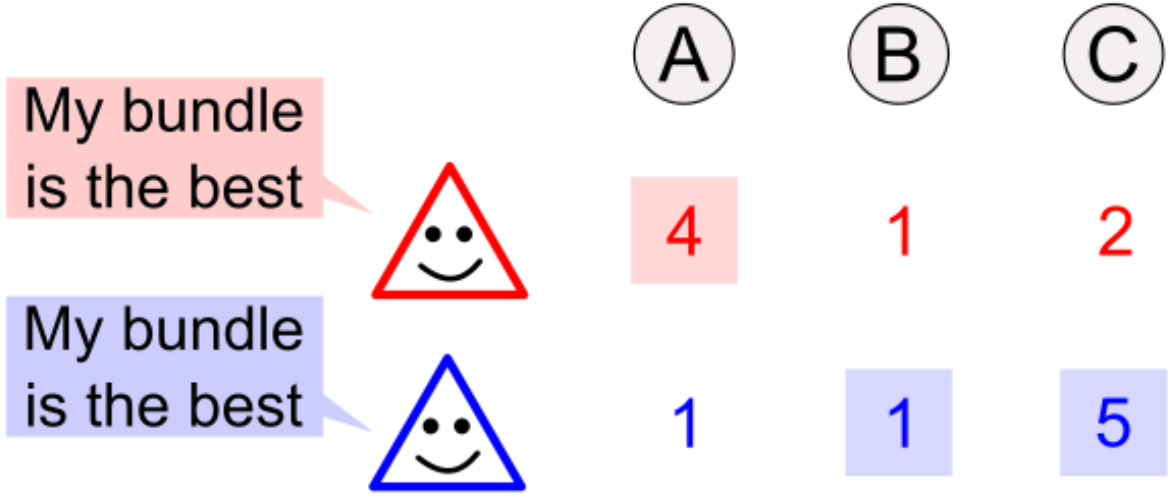

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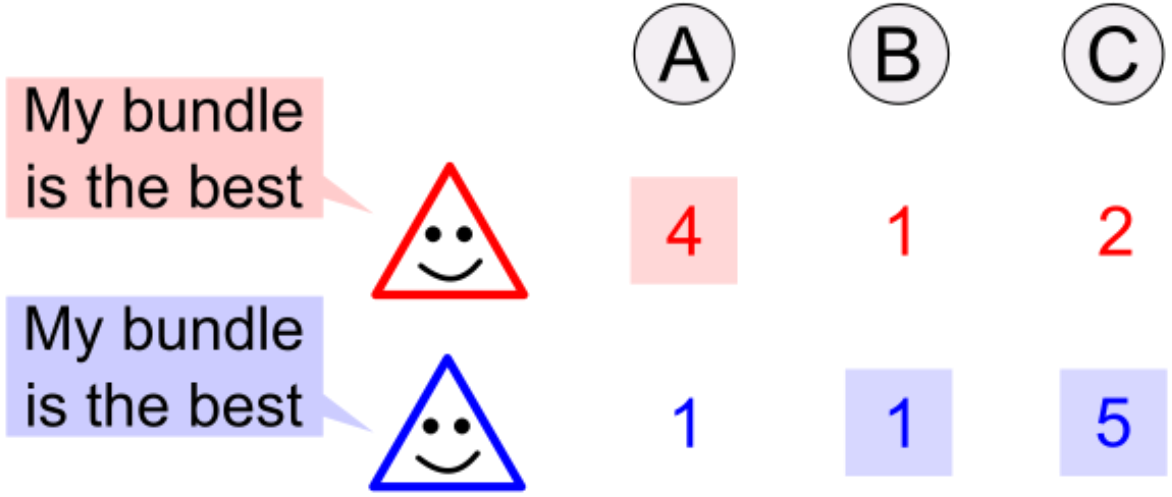

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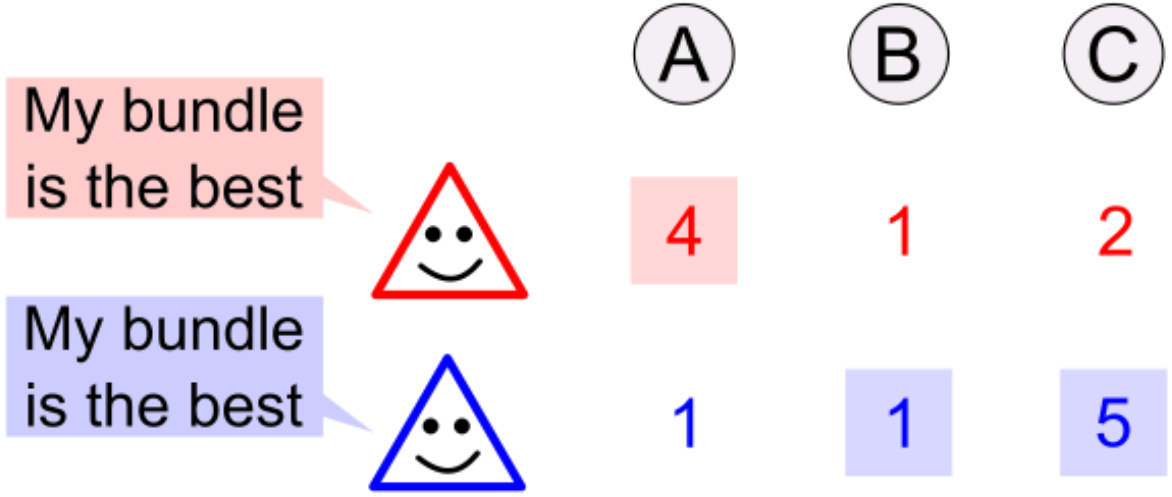

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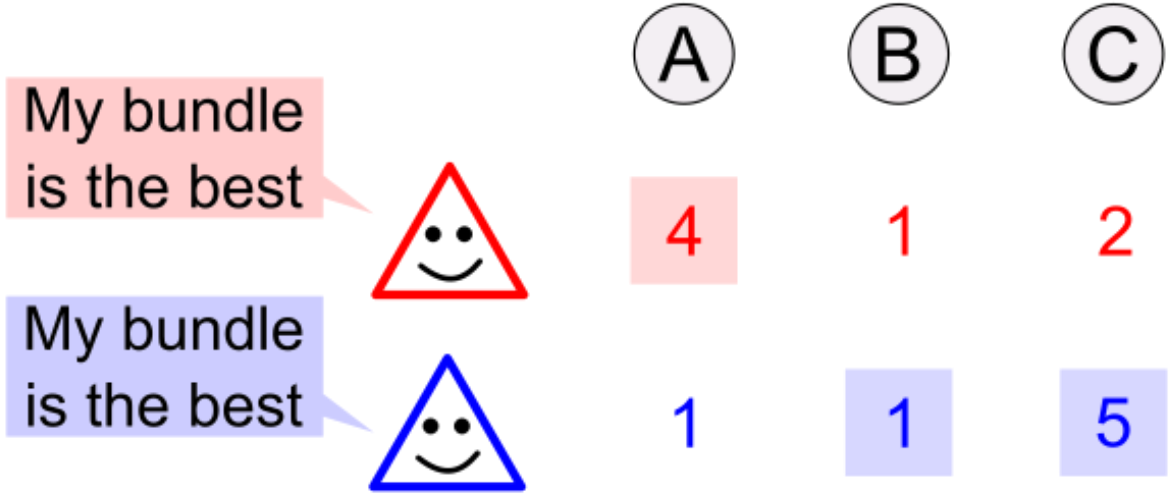

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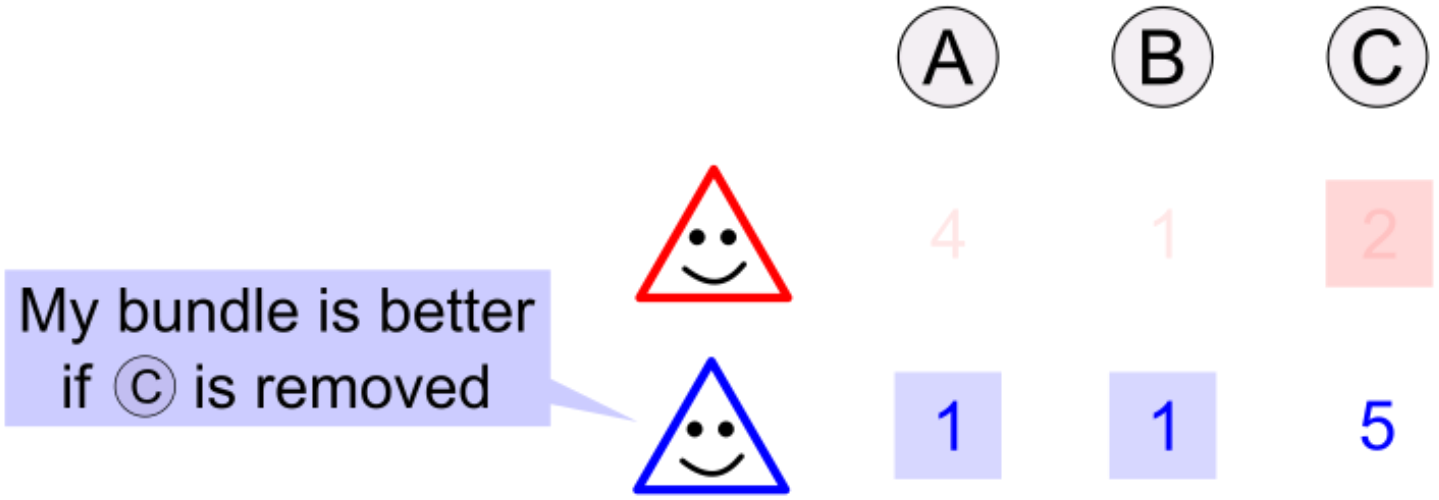

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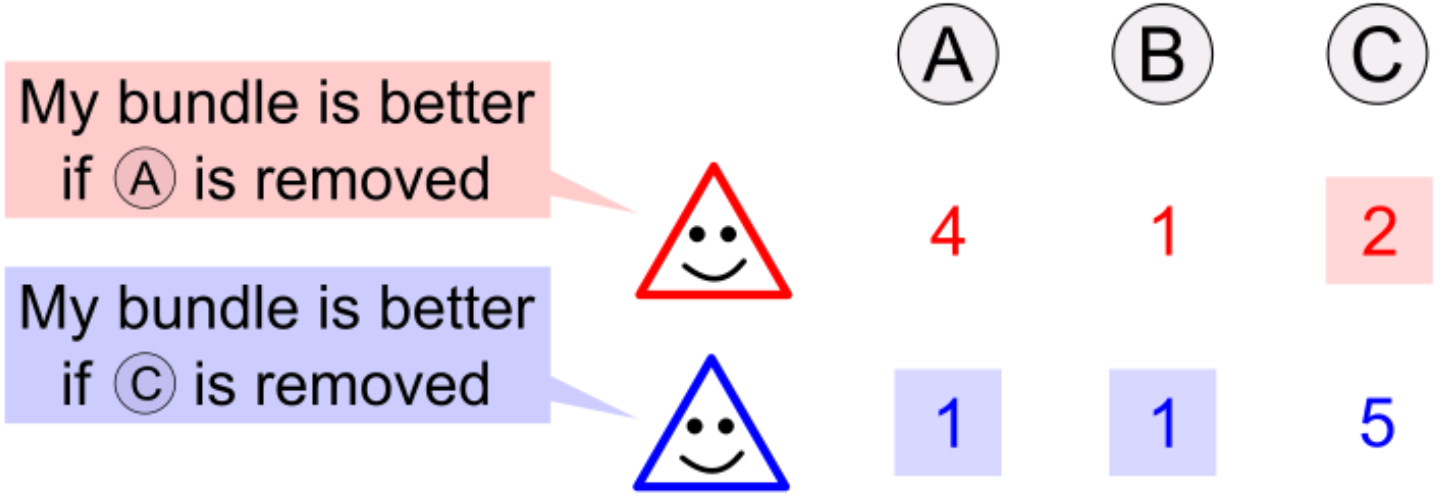

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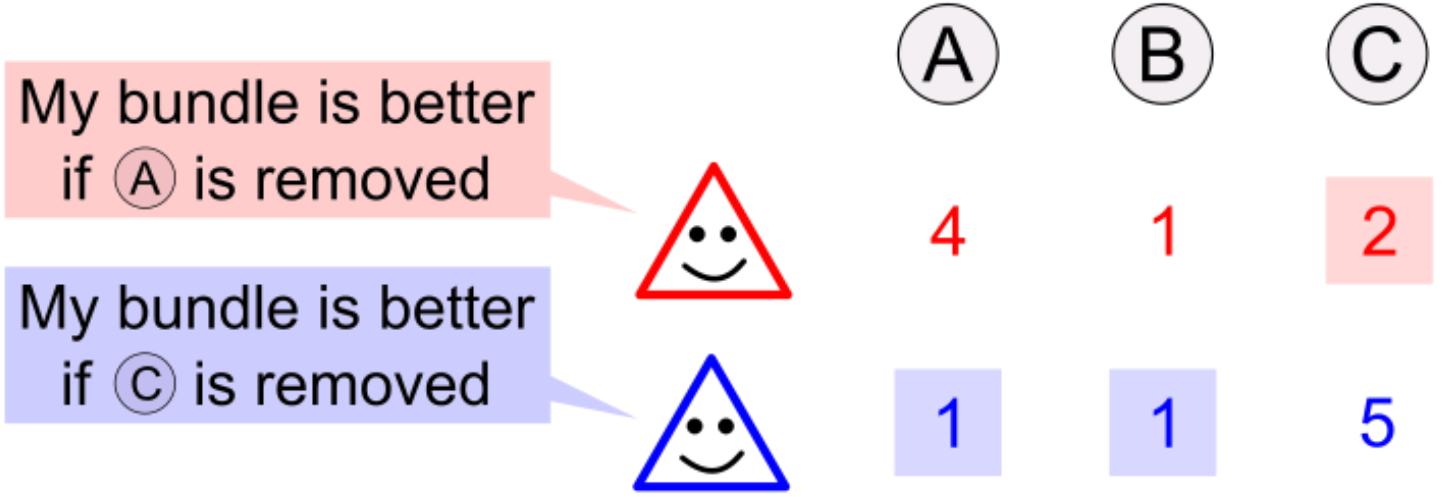

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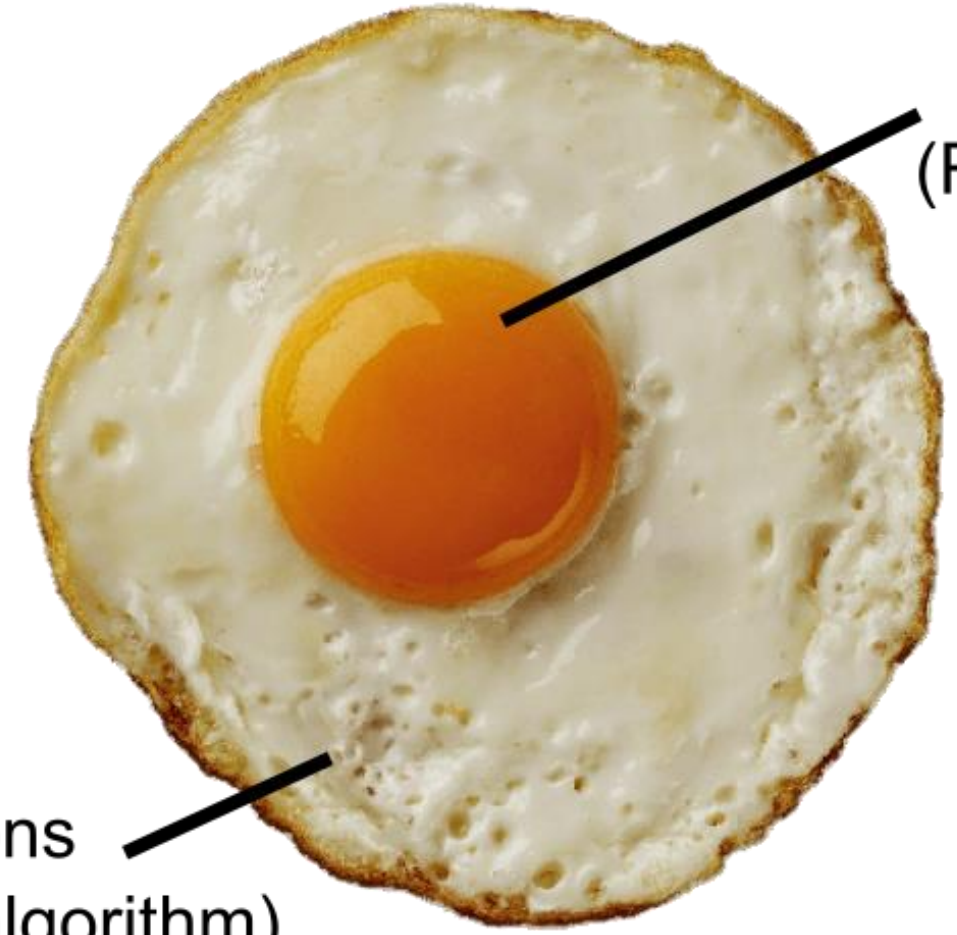
Guaranteed to exist and efficiently computable

Coming Up

Algorithms for finding an EF1 allocation



Additive valuations
(Round-robin algorithm)

Monotone valuations
(Envy-cycle elimination algorithm)

Round-robin algorithm

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First round

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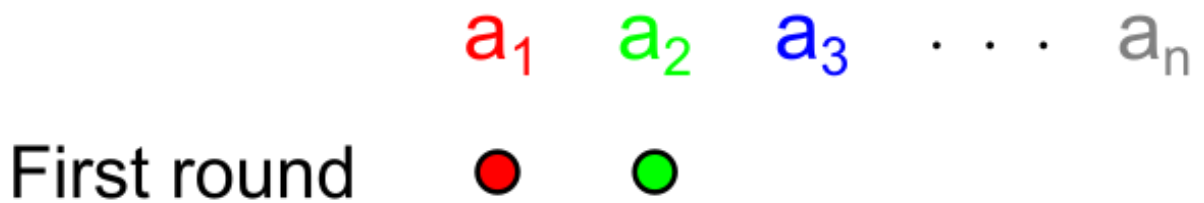

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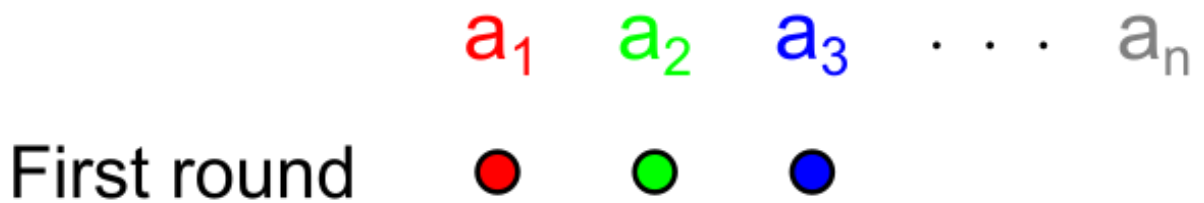

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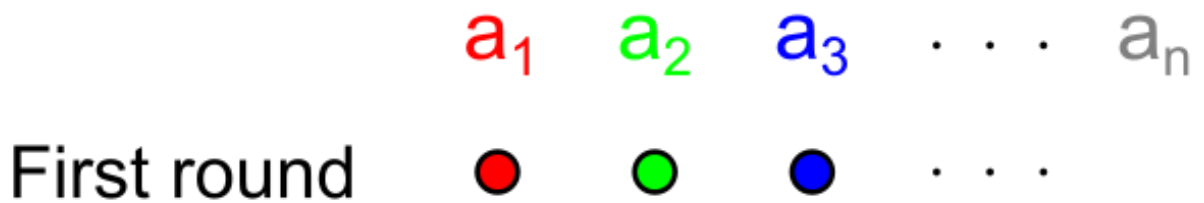

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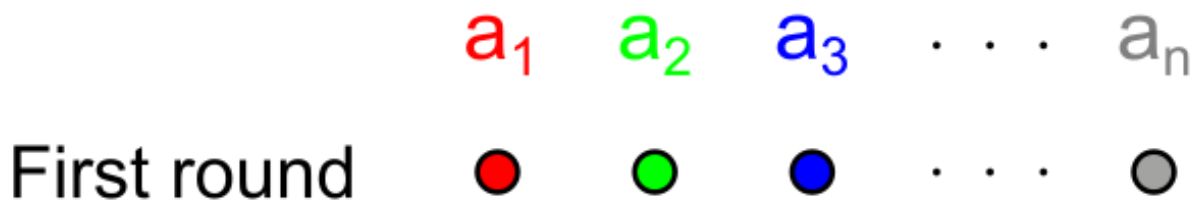

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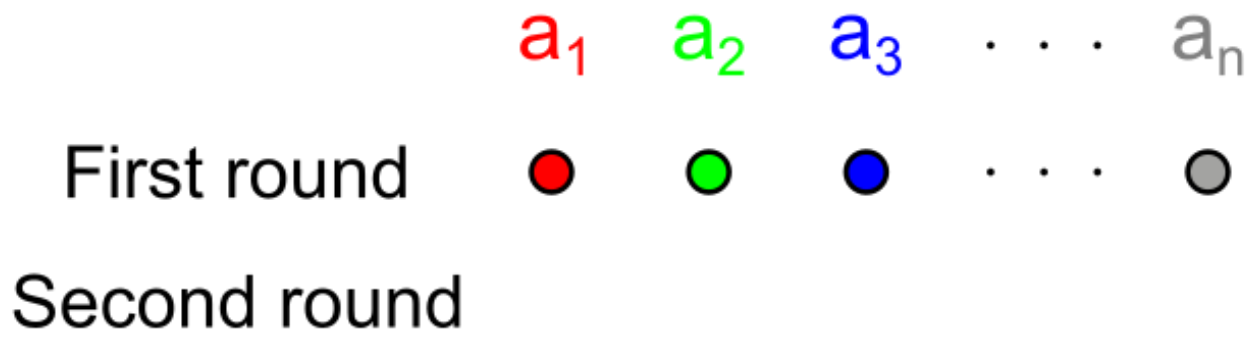

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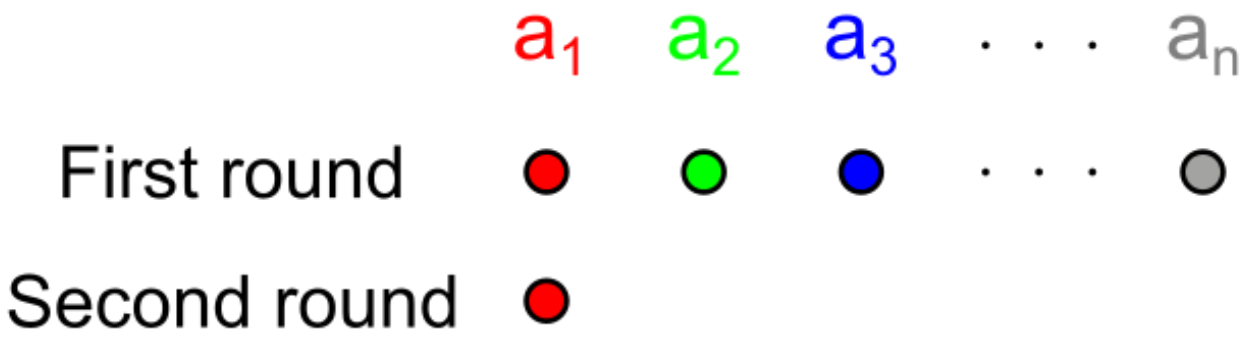

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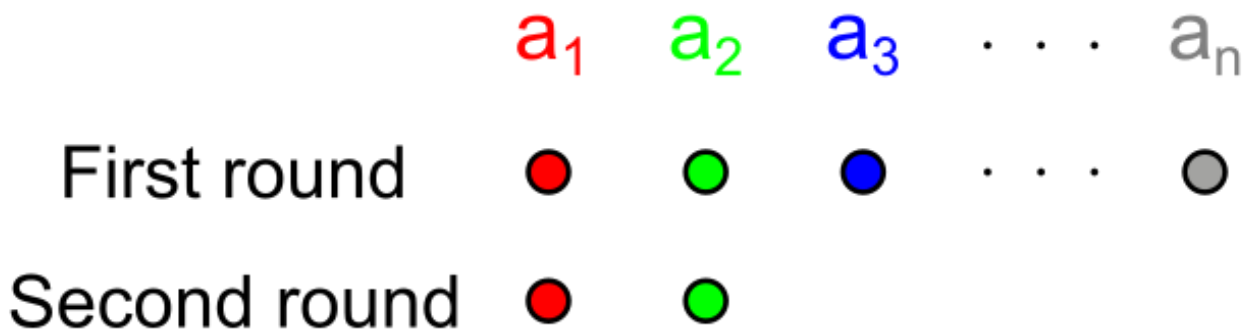

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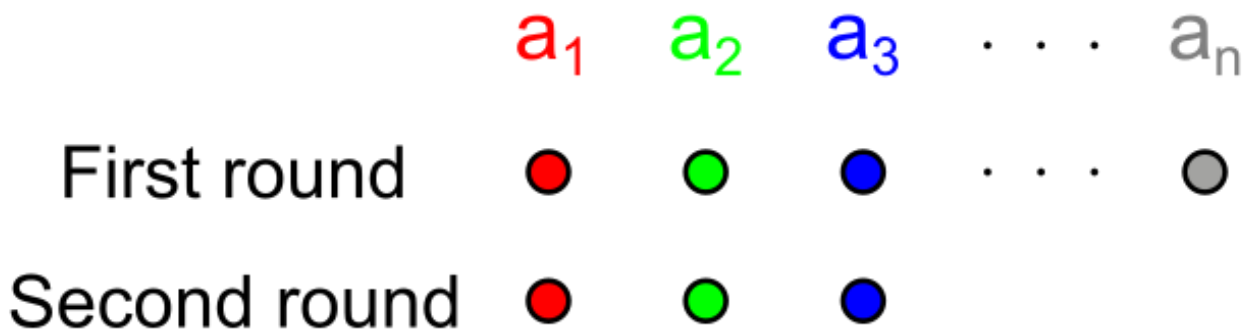

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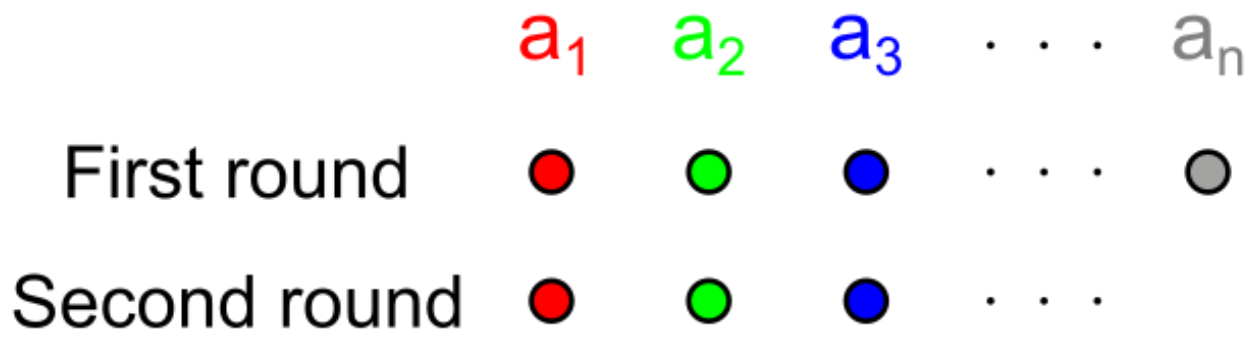

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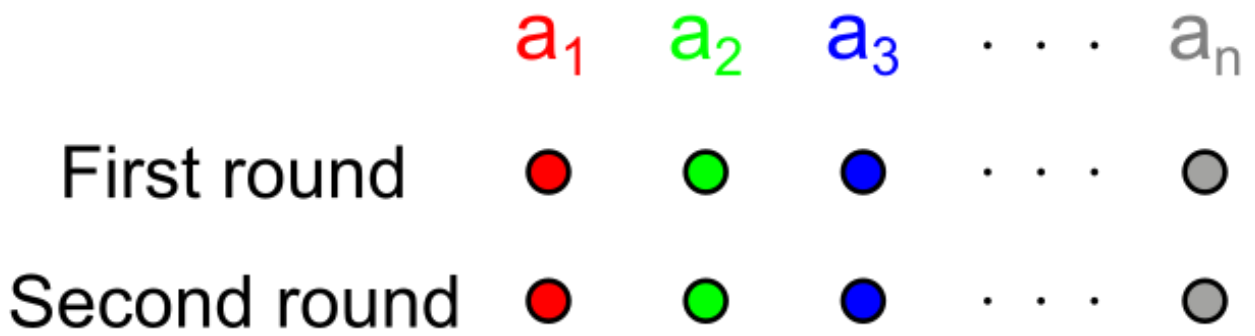

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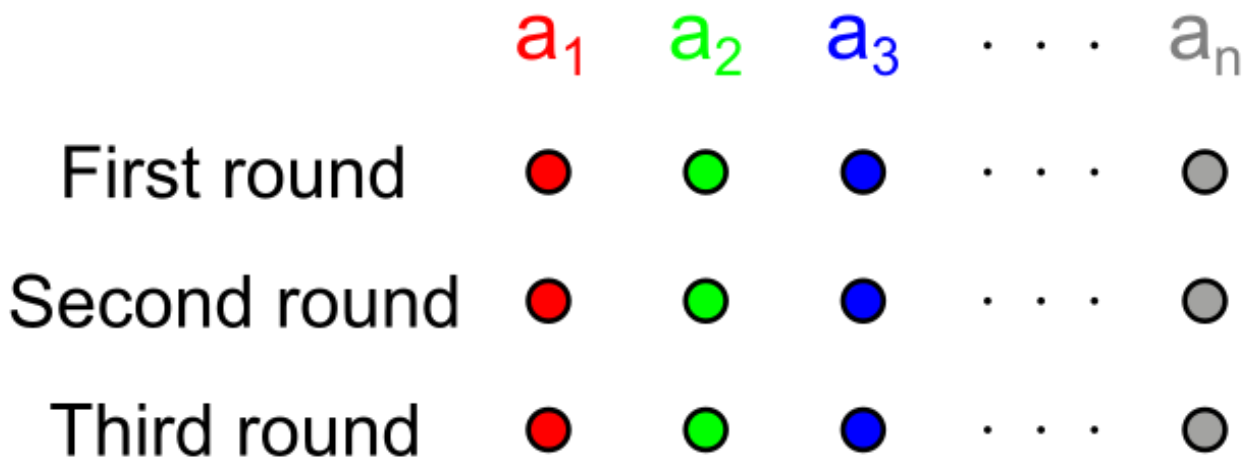

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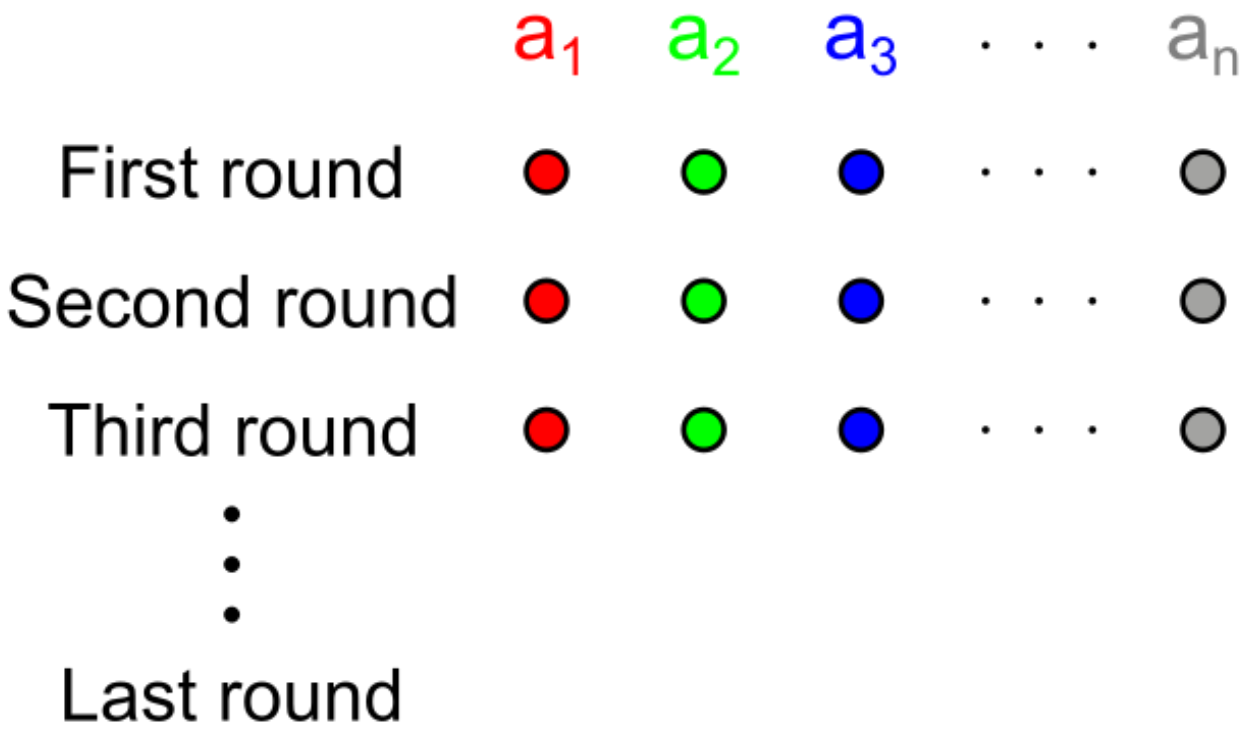

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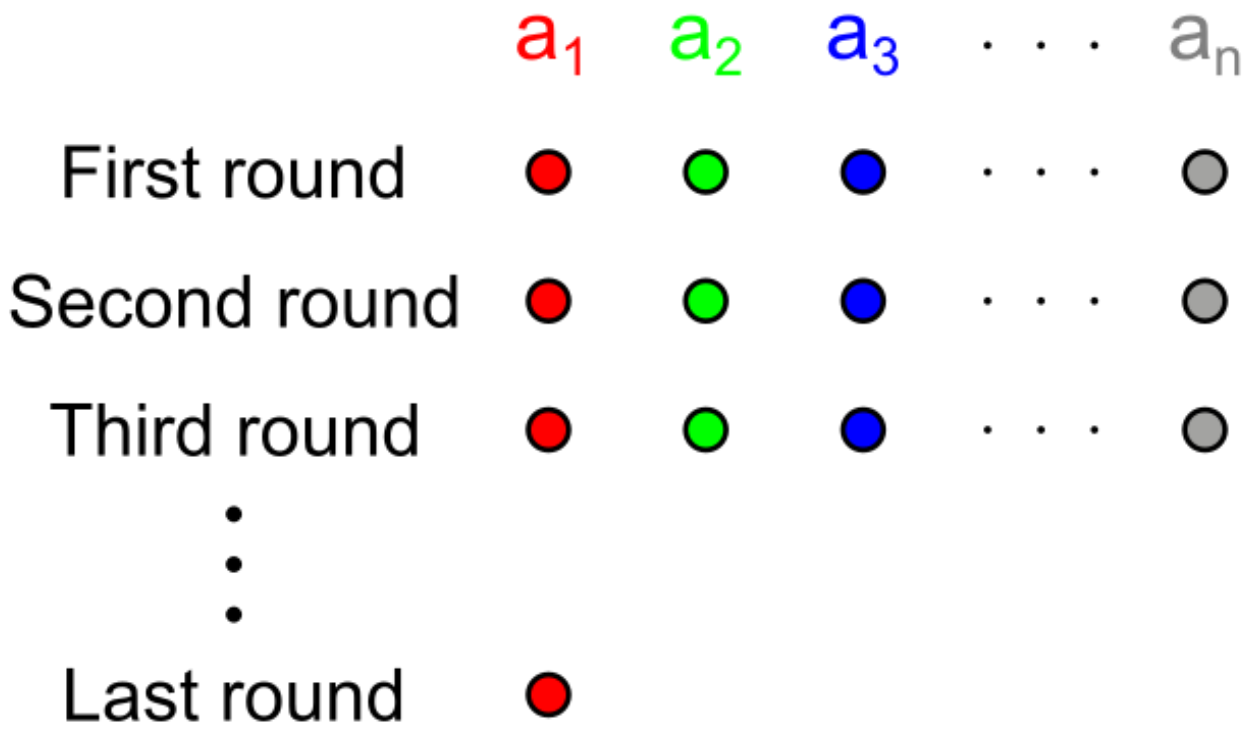

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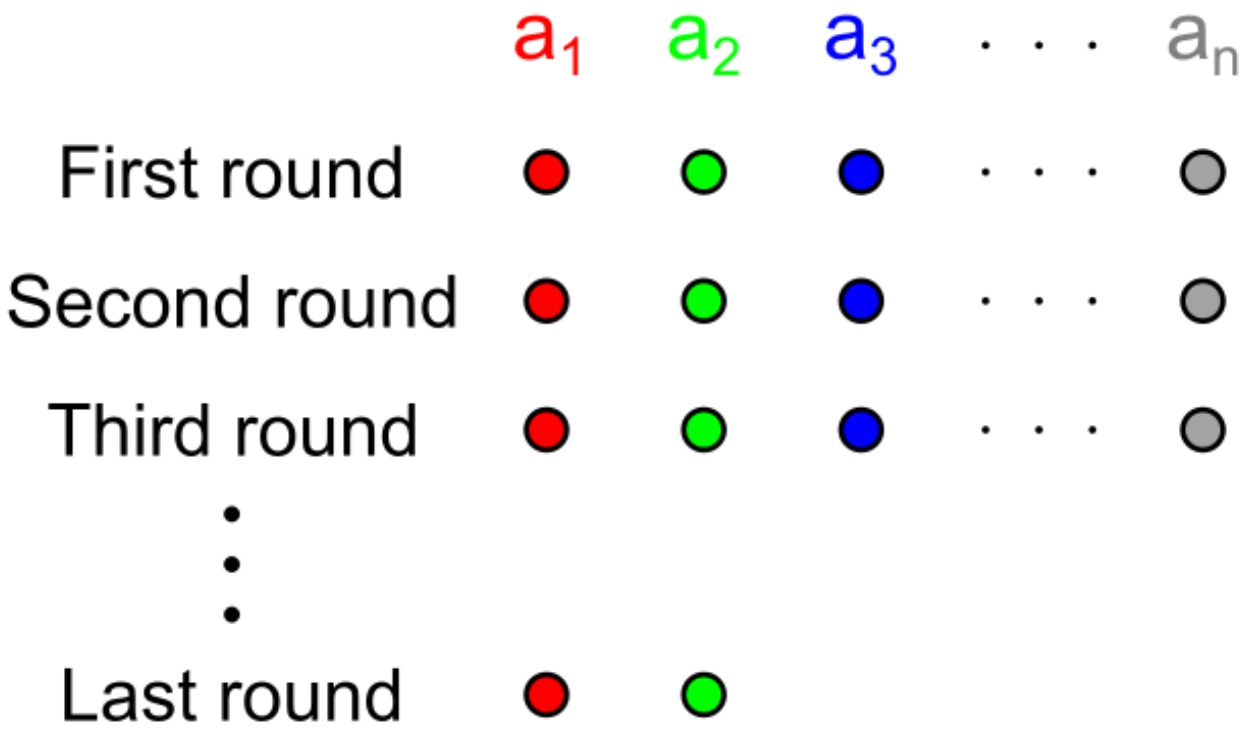

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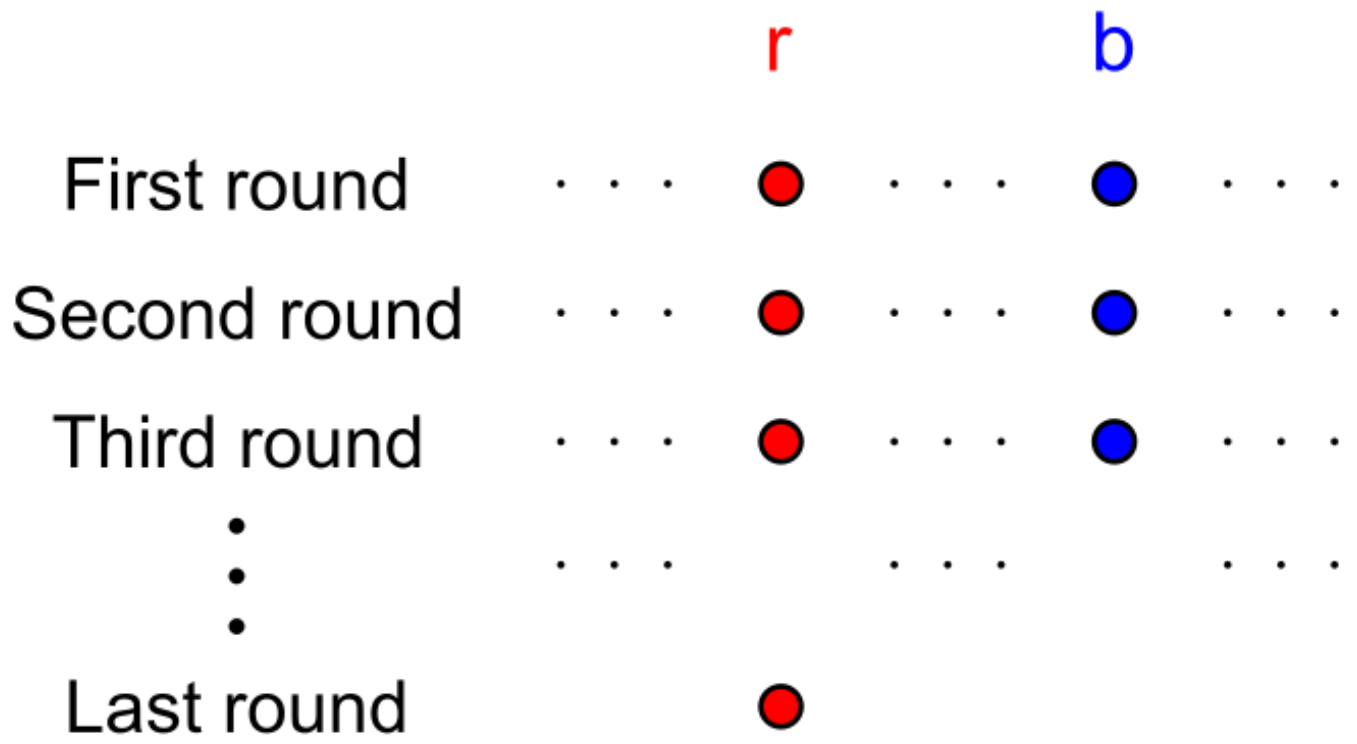

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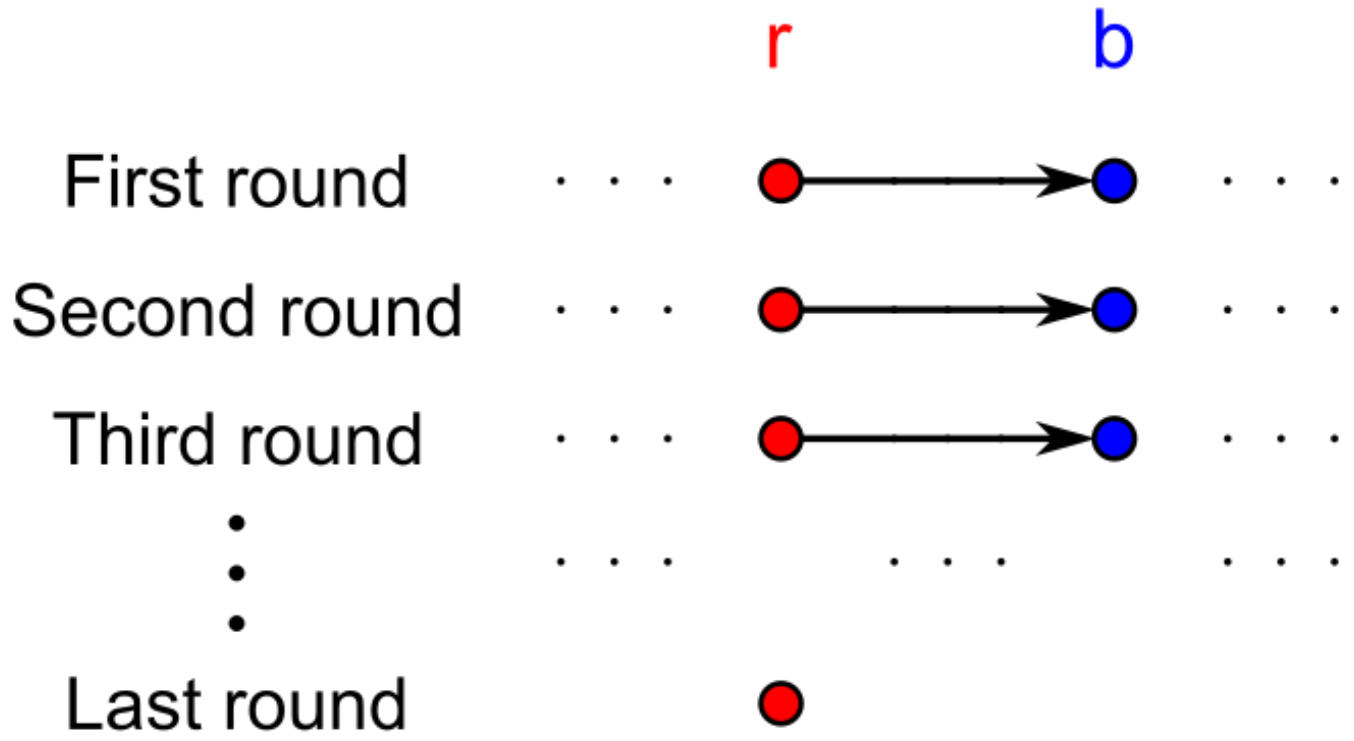

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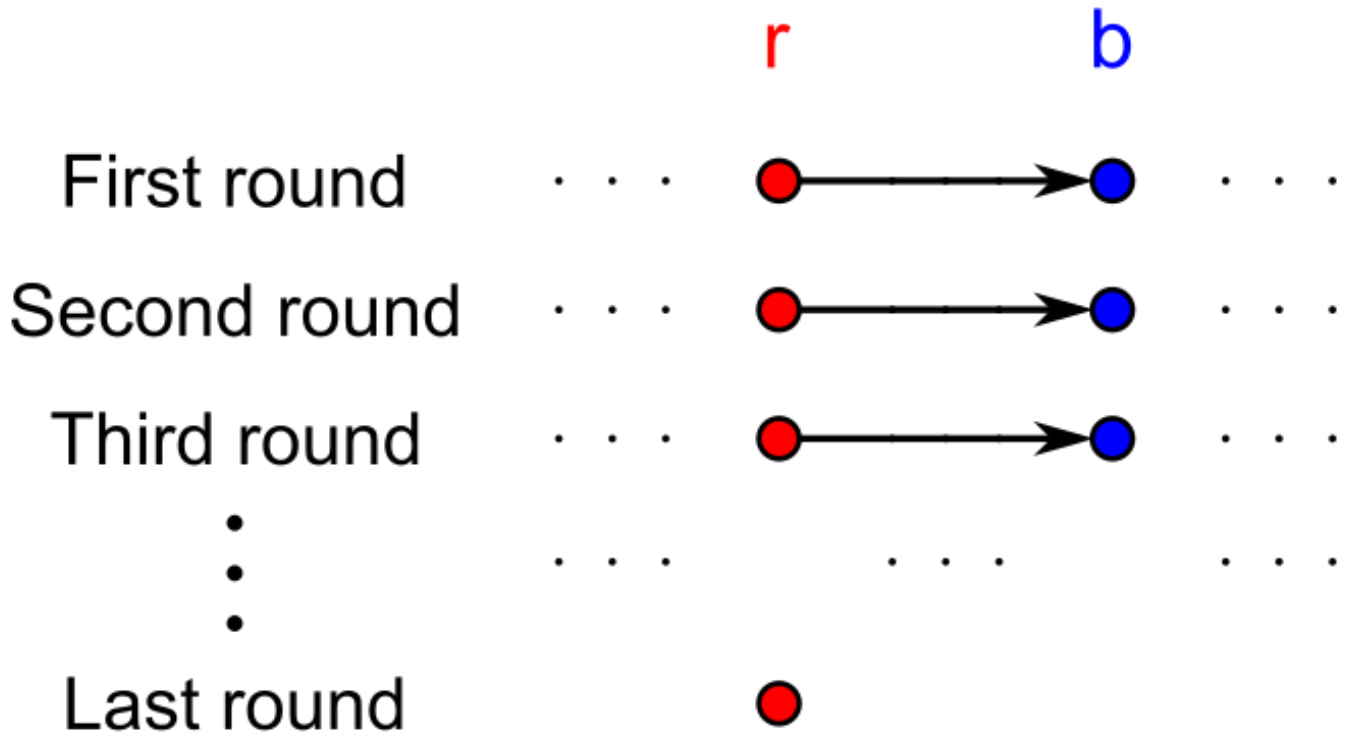

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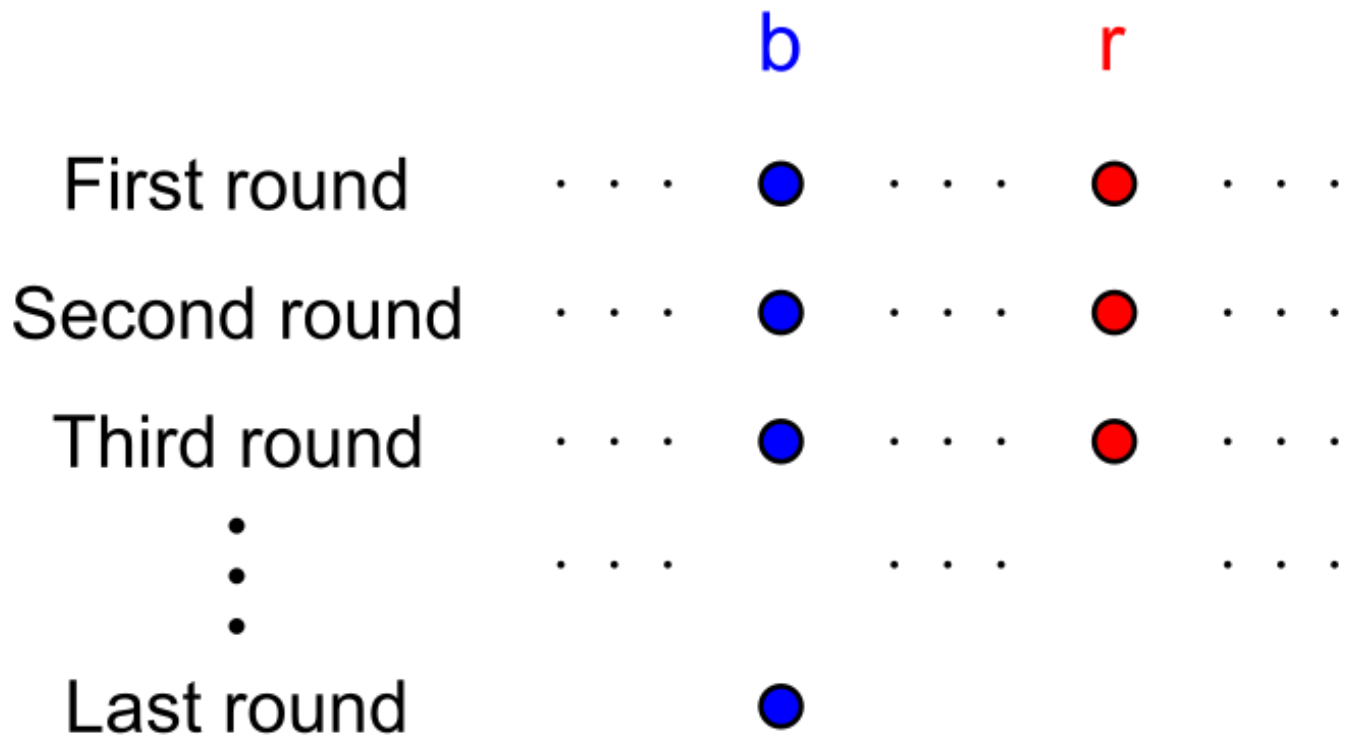

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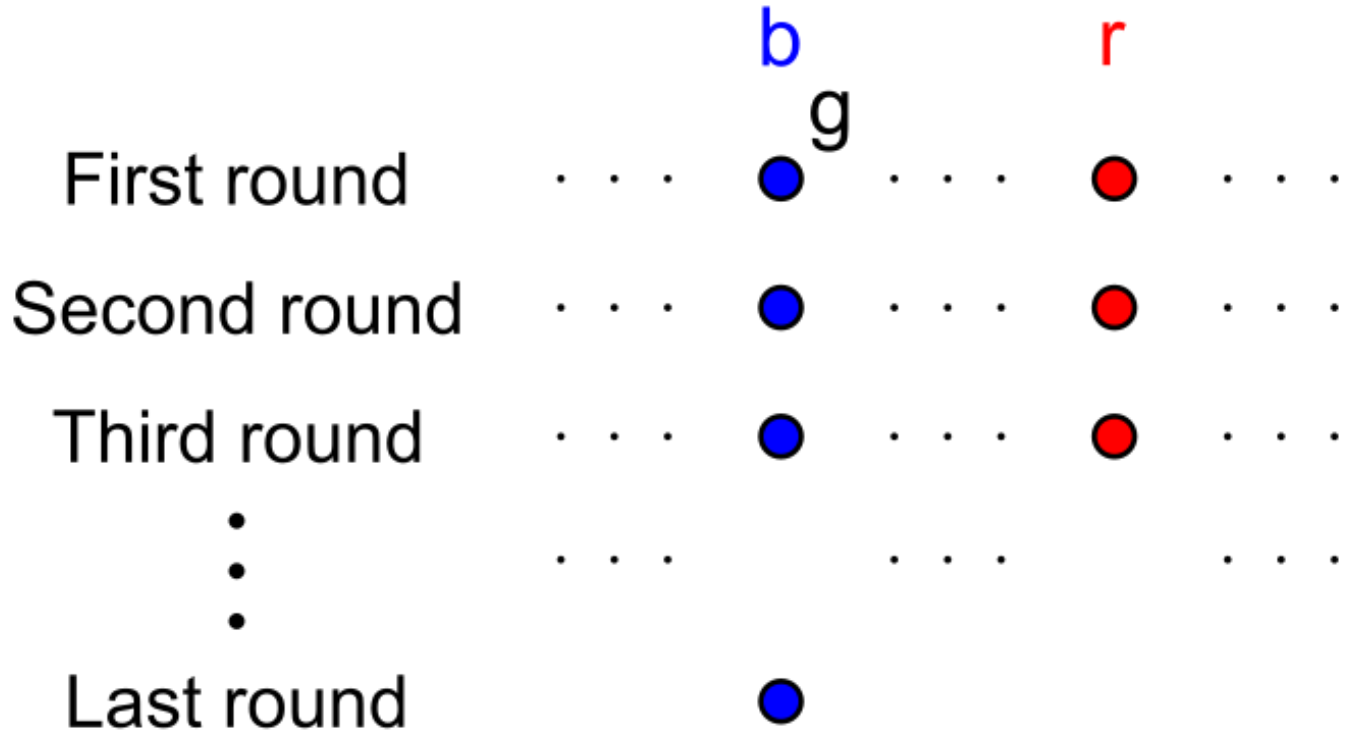

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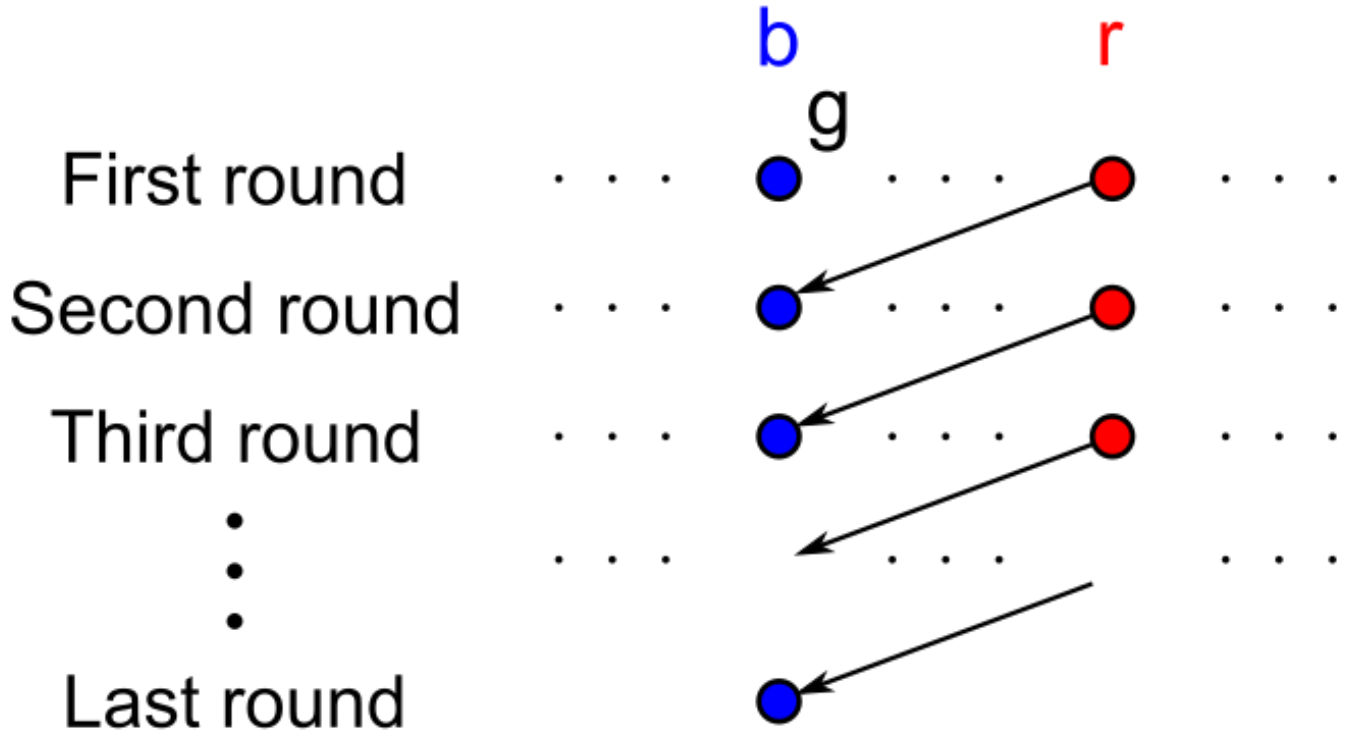

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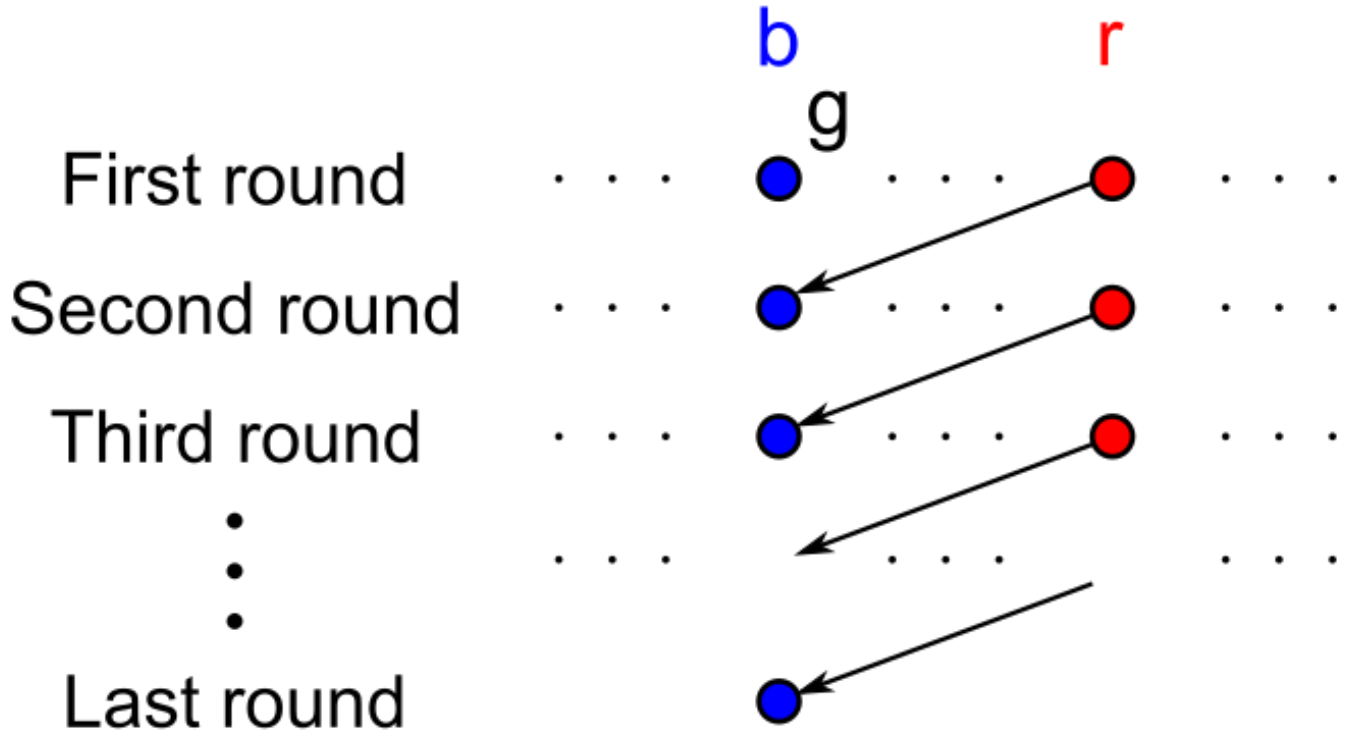

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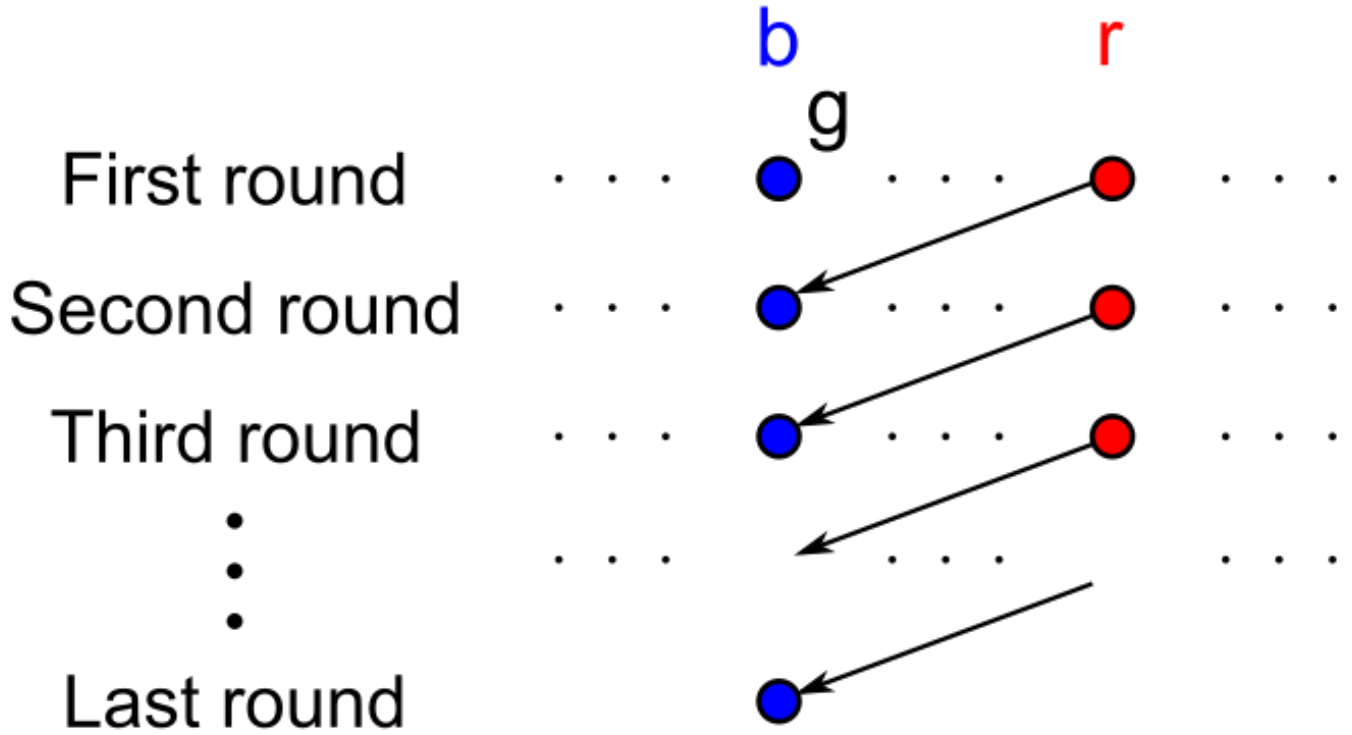

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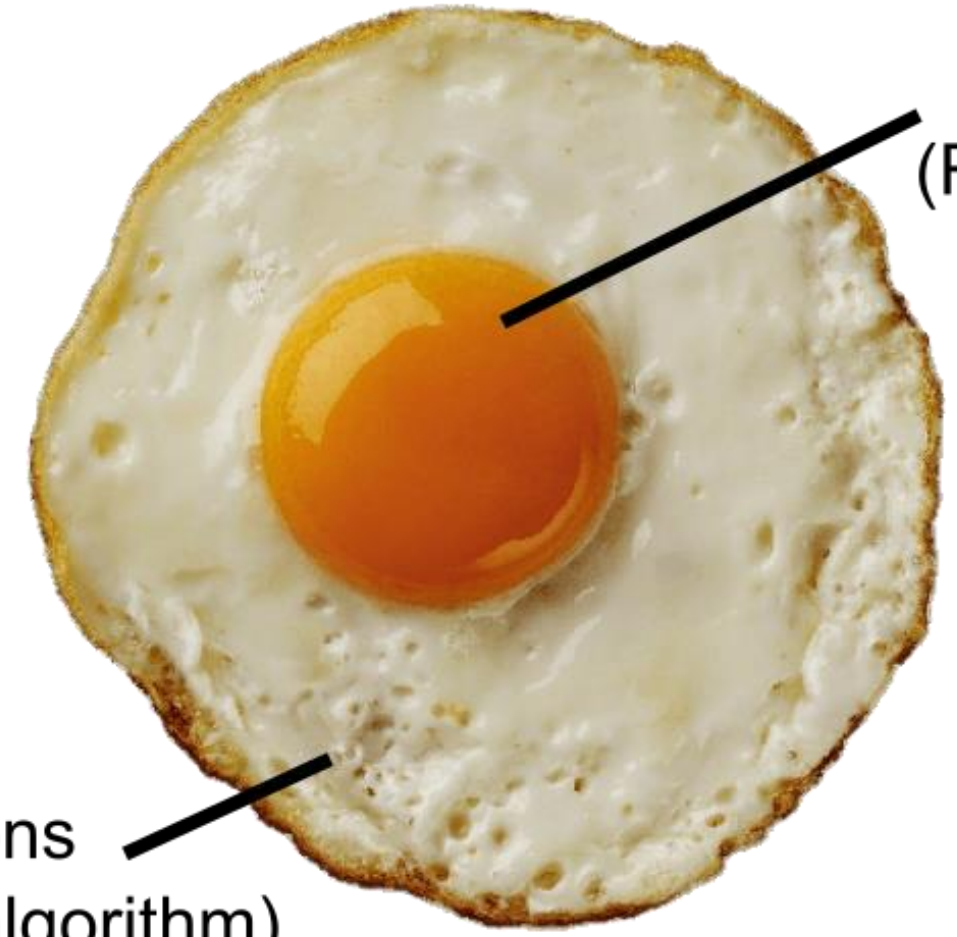
Algorithms for EF1



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Envy graph of an allocation

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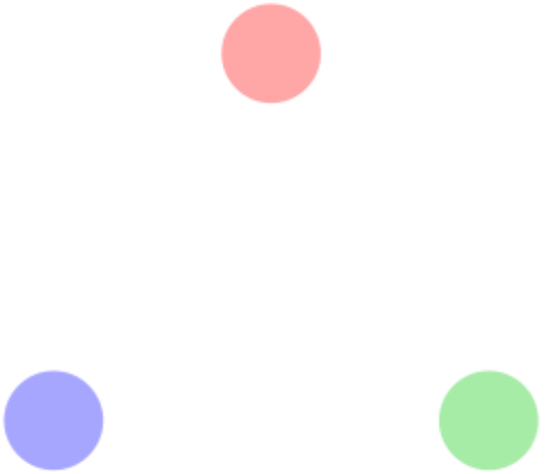

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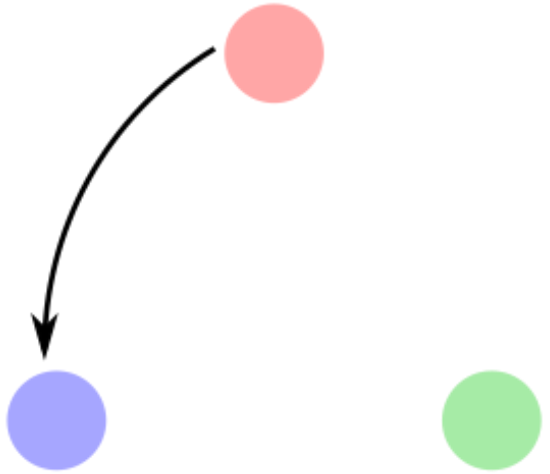

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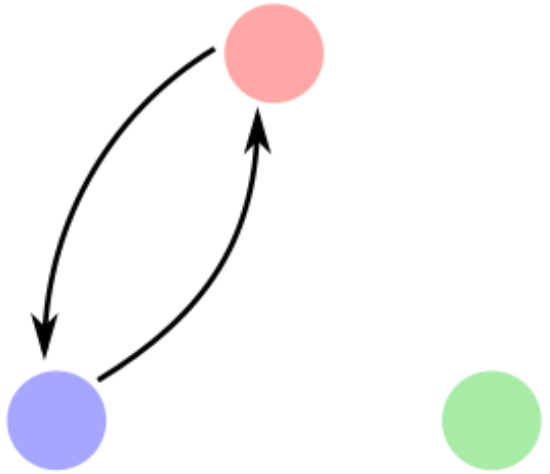

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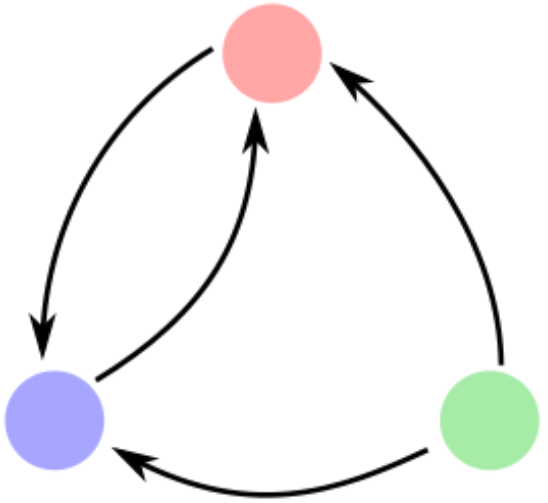

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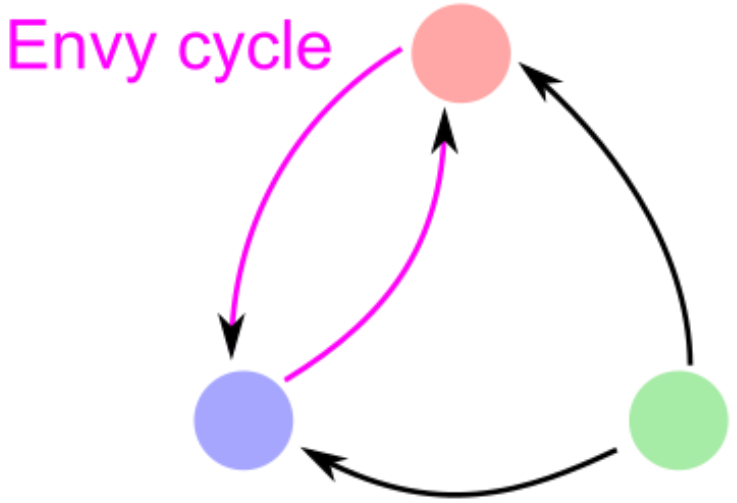

Envy-cycle elimination algorithm

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each agent in the cycle gets the bundle that it is pointing to

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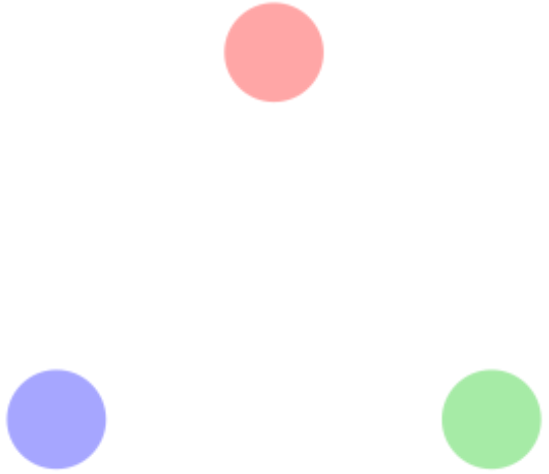

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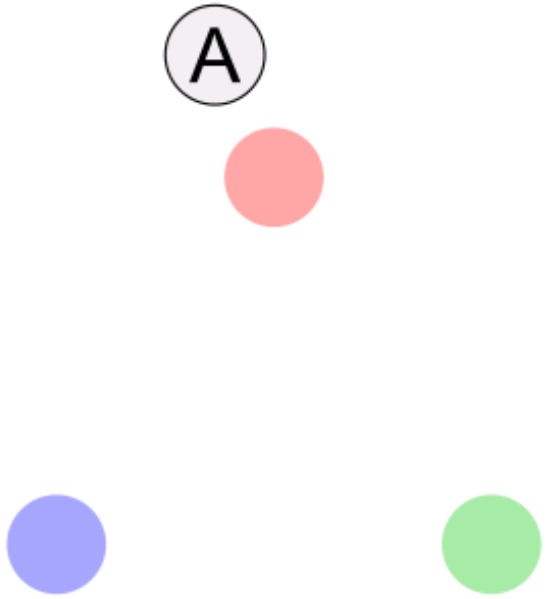

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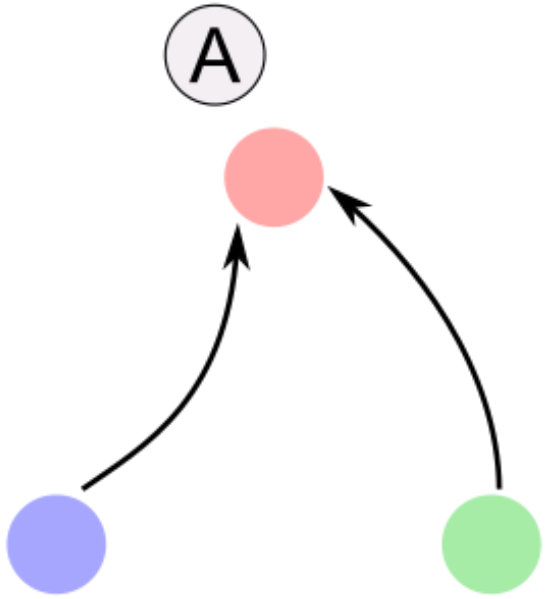

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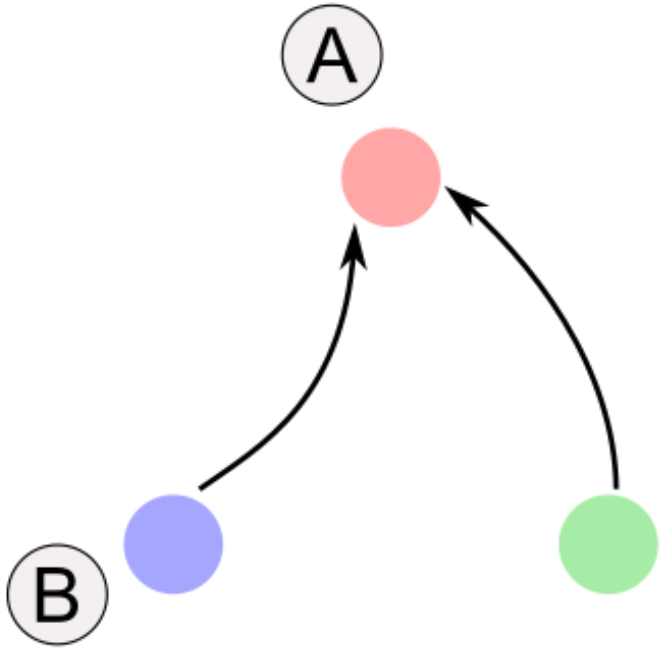

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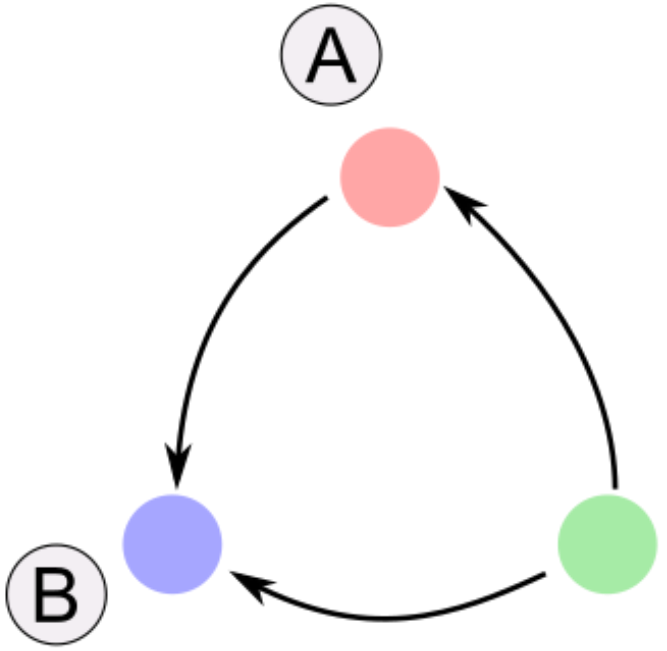

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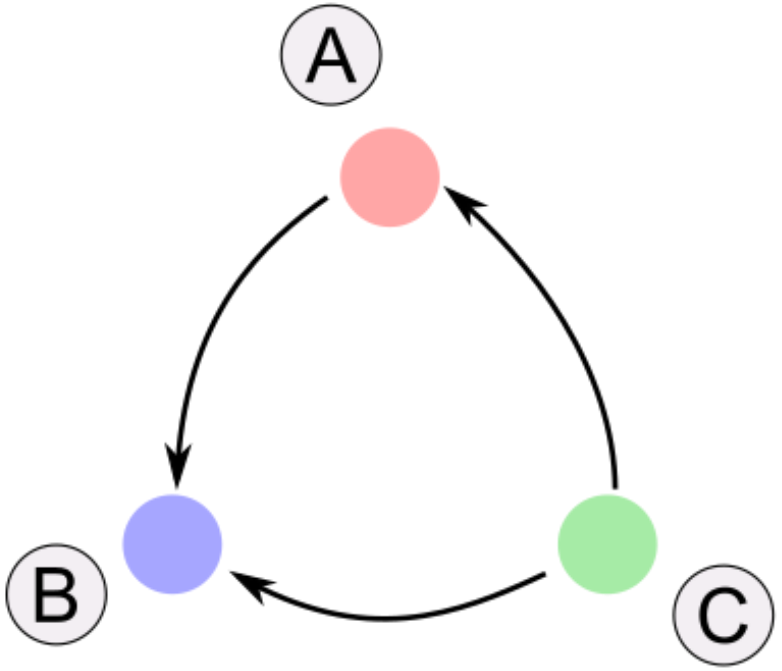

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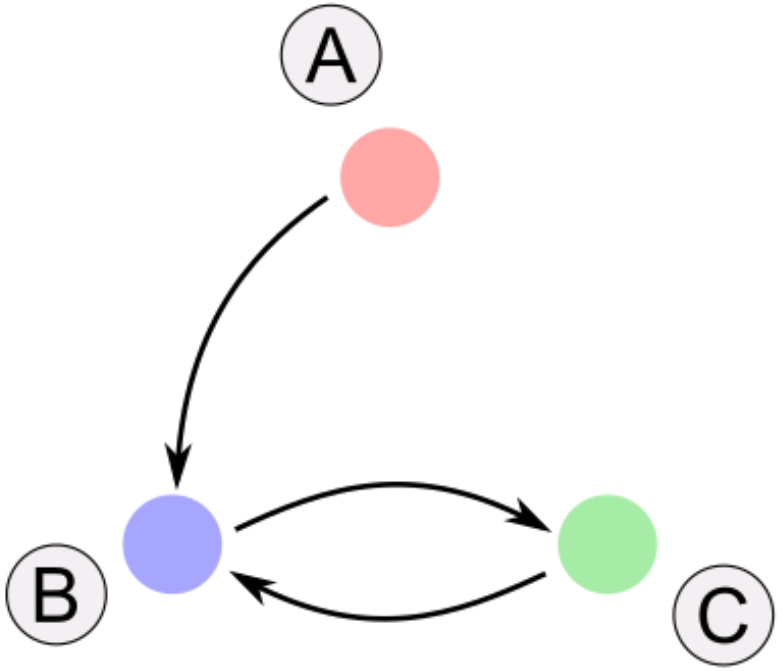

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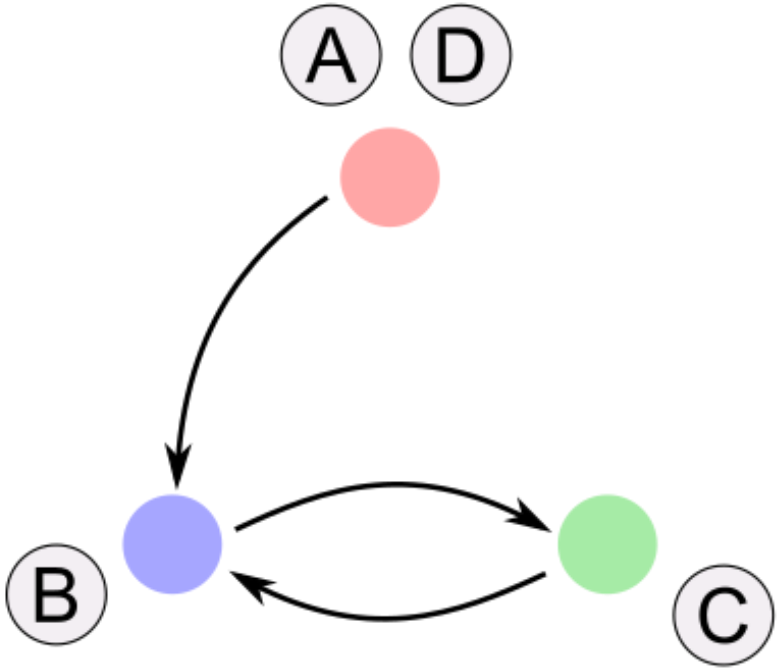

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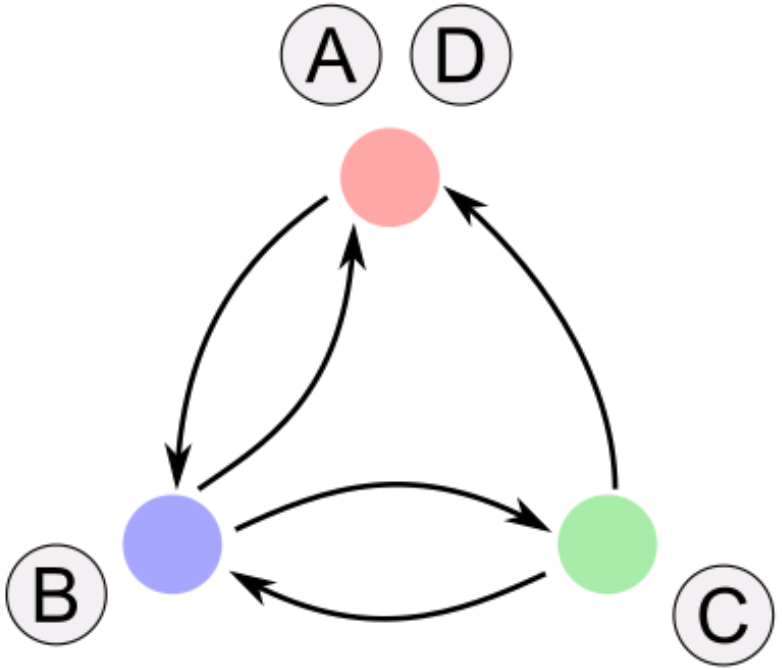

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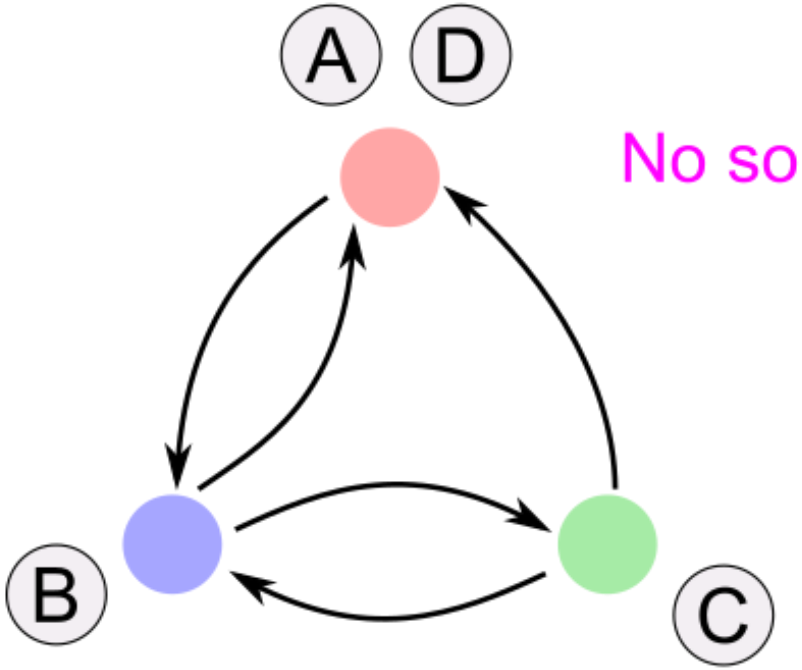

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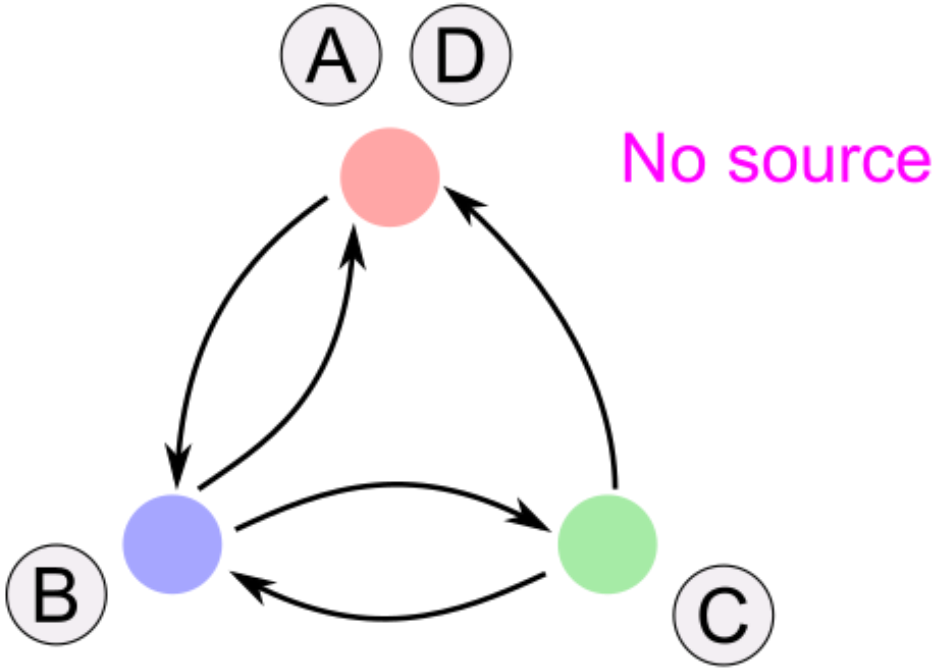

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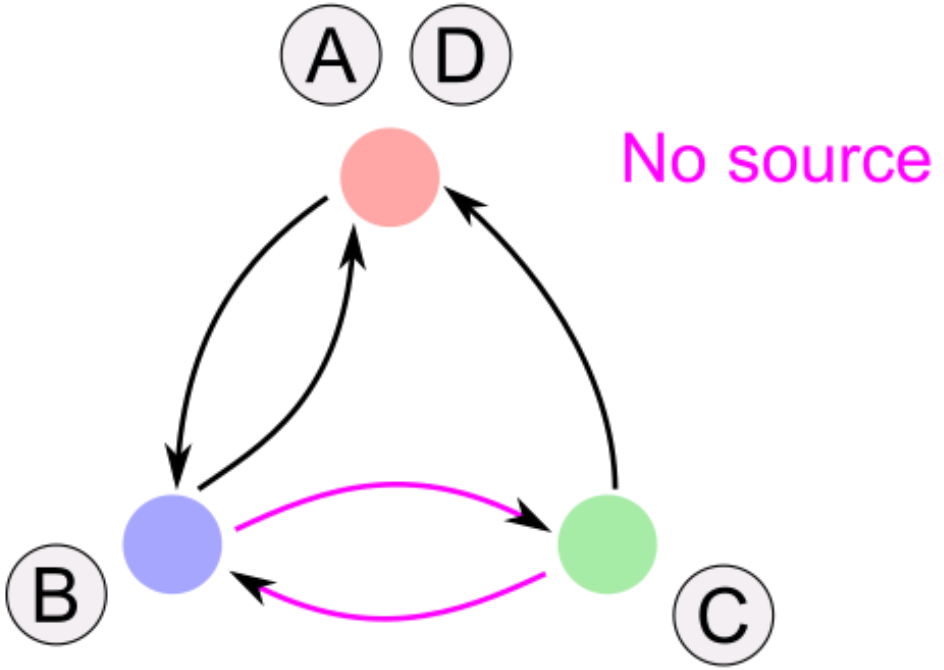

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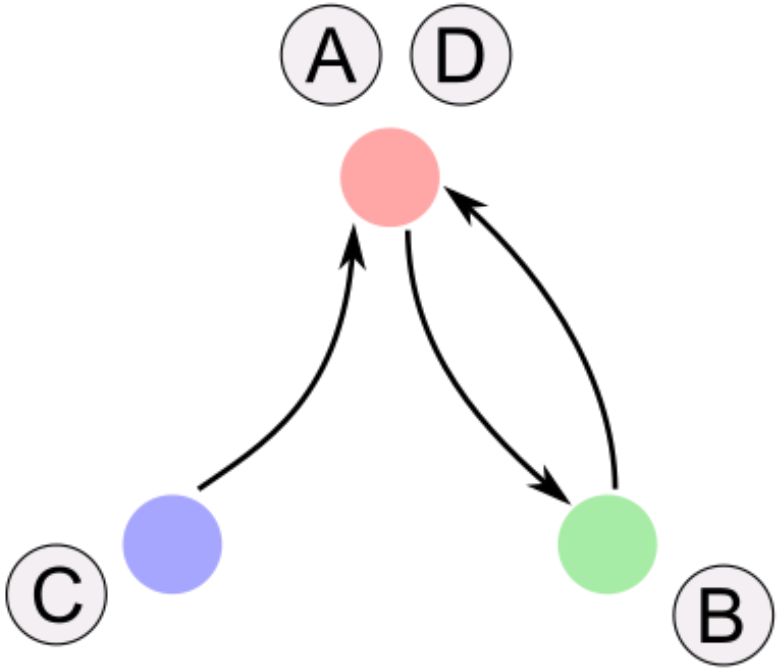

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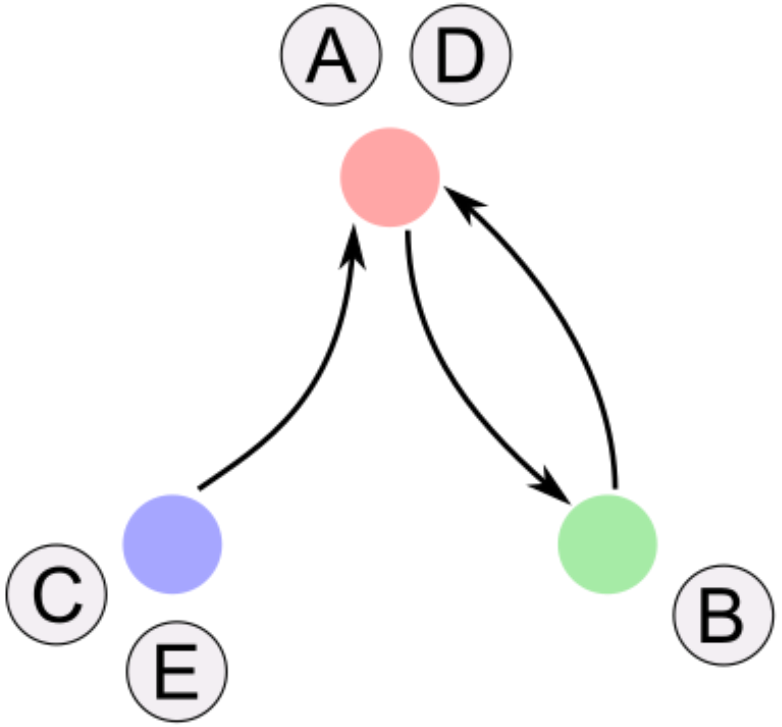

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1 Does the algorithm terminate?

2 Does the algorithm terminate in polynomial time?

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- With n agents, at most $O(n^2)$ cycle resolutions required to create a source.
- Polynomial running time!

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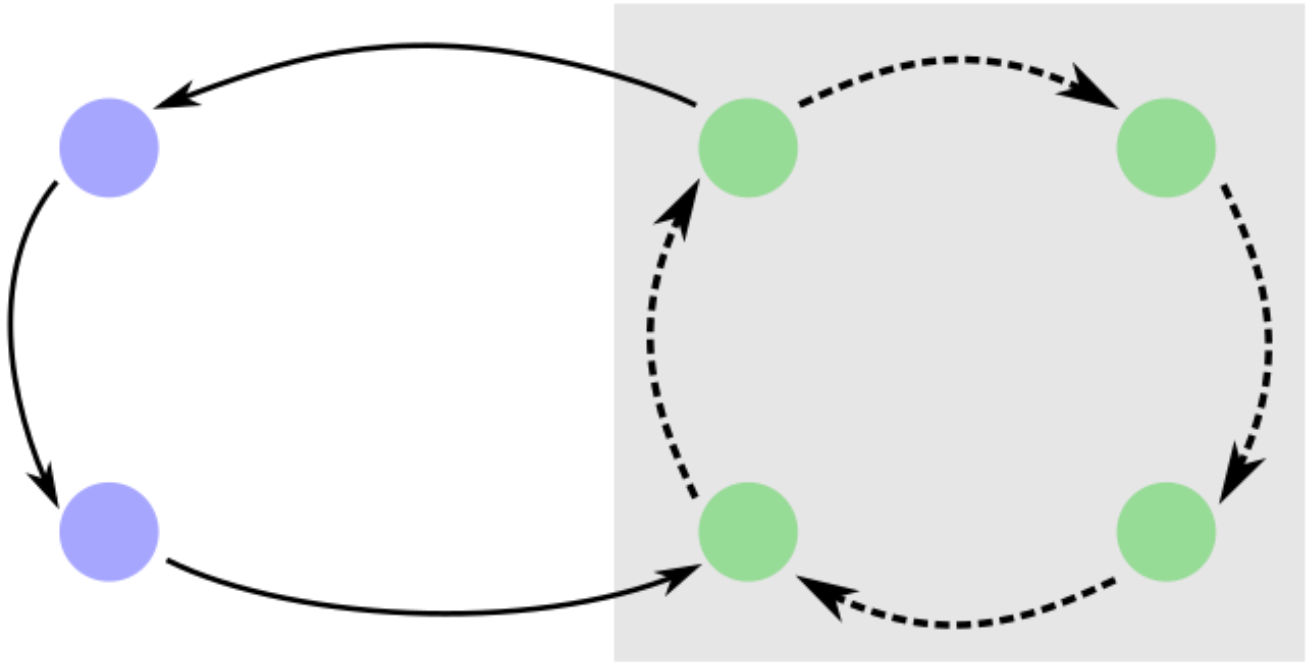

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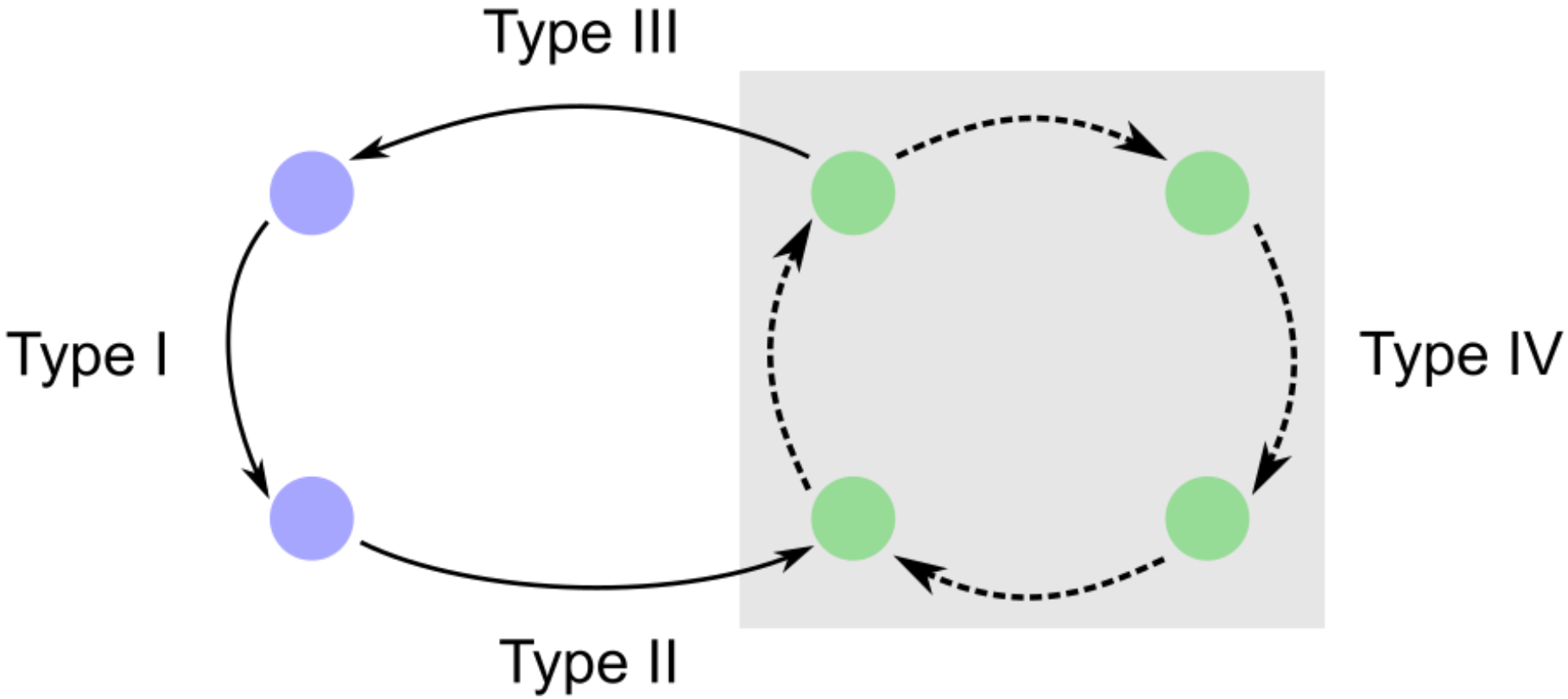

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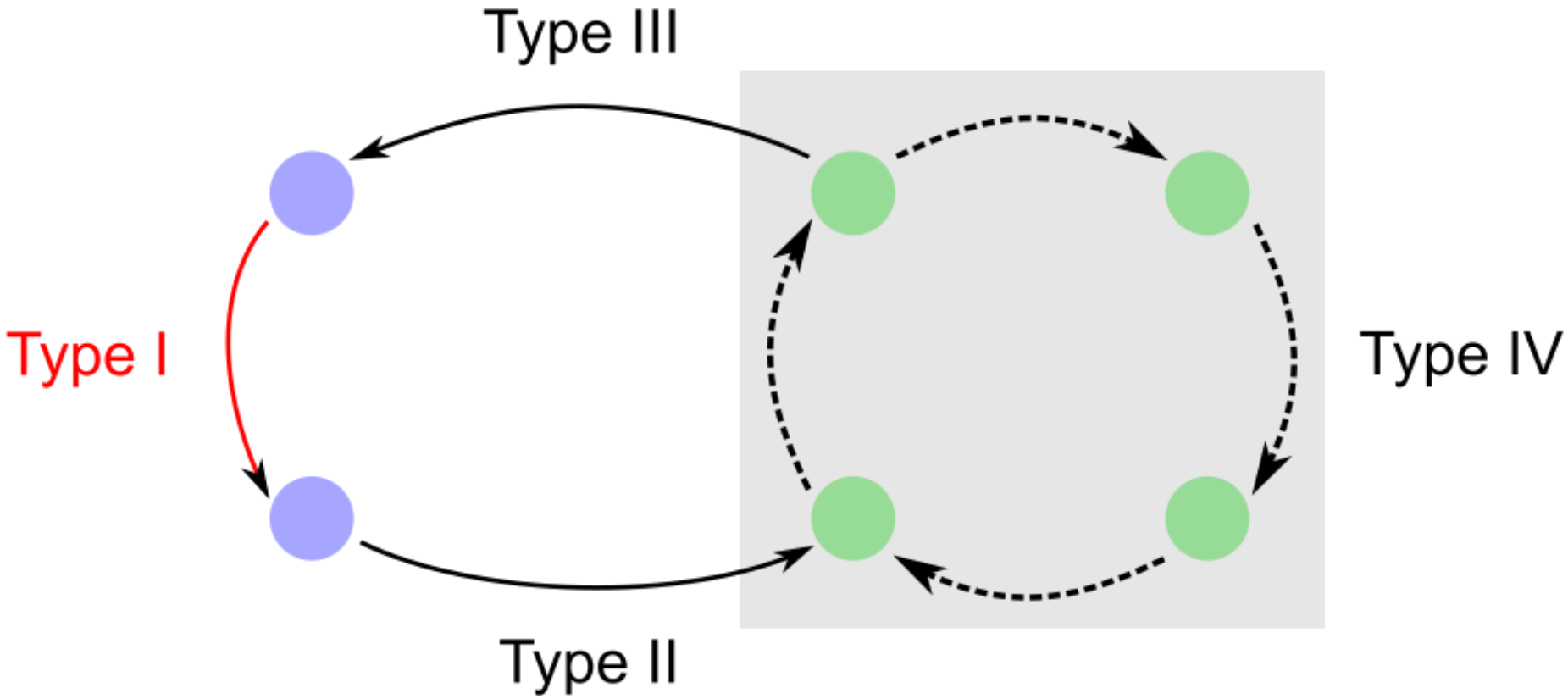

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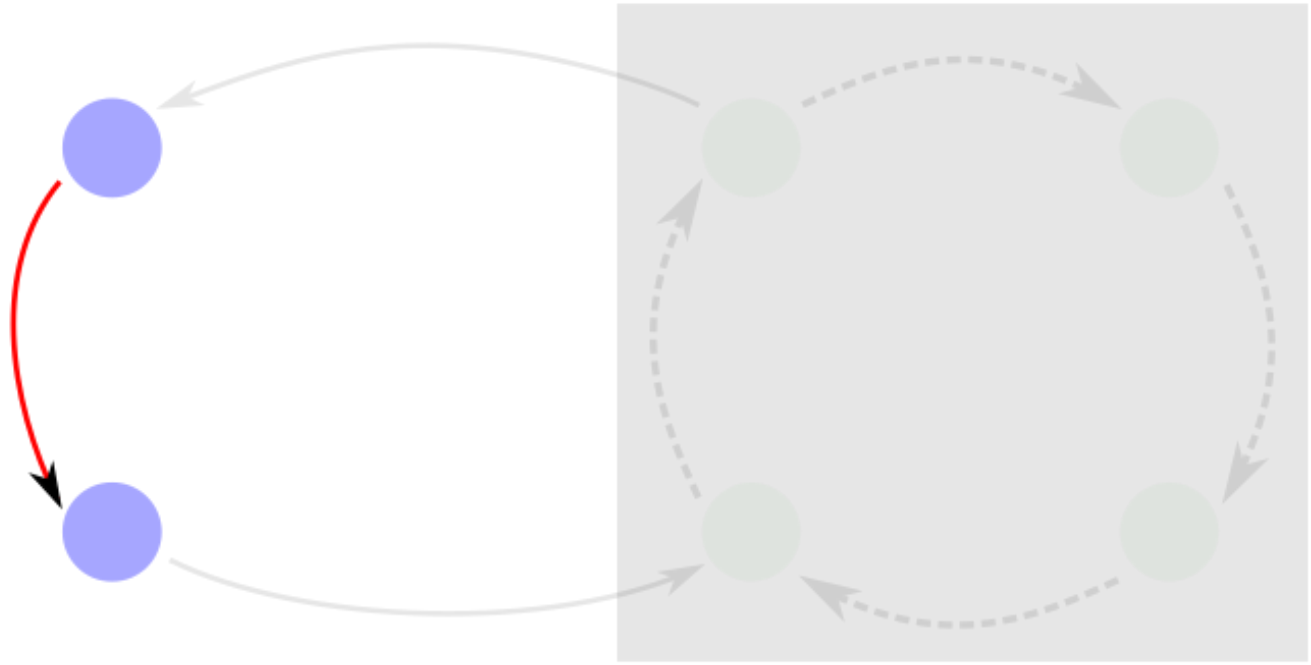

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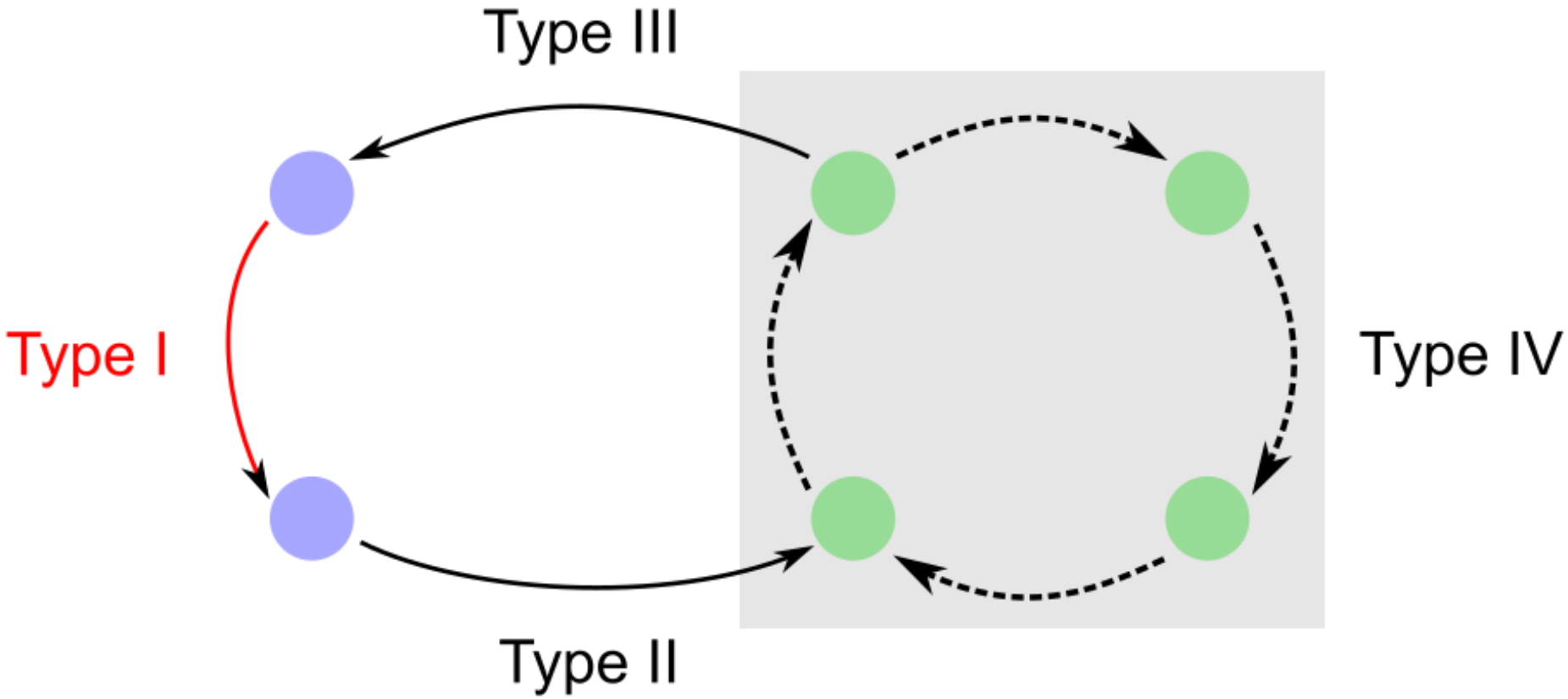

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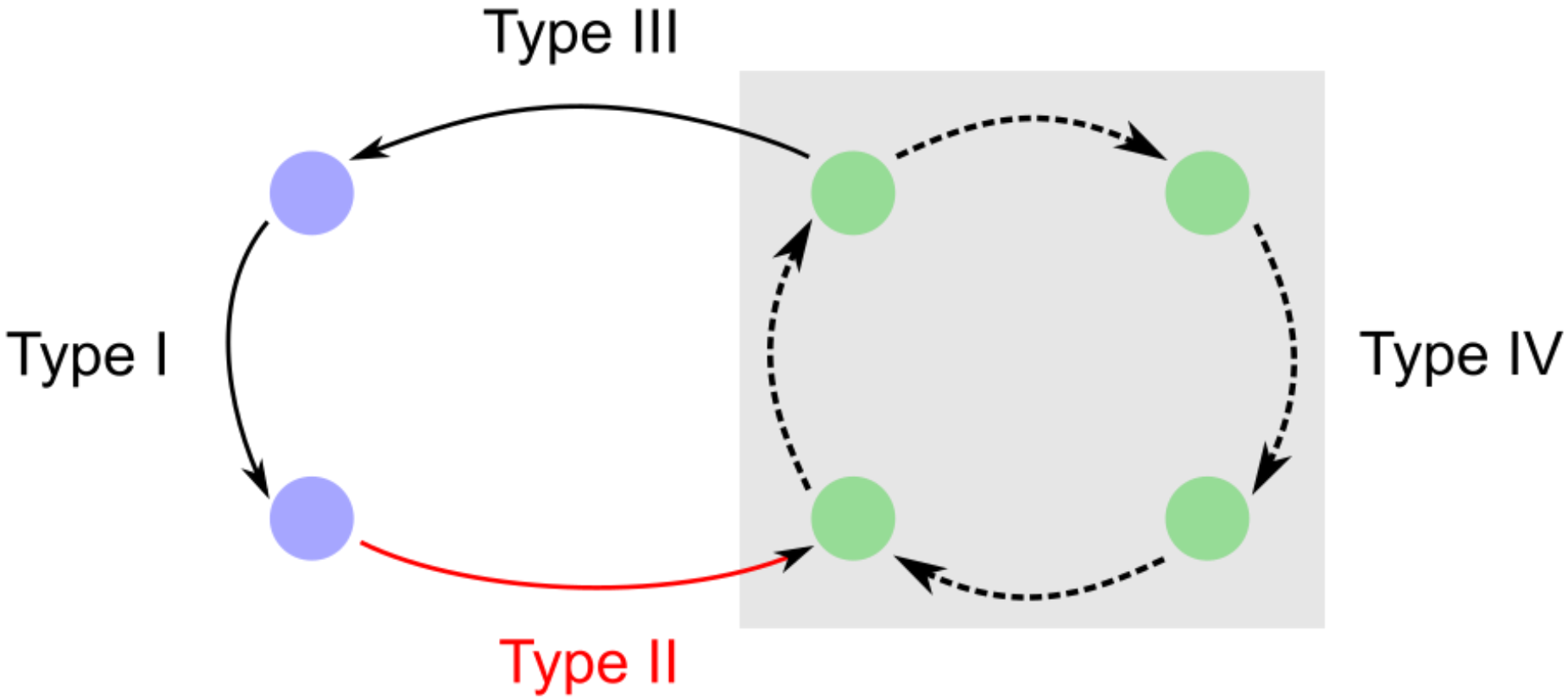

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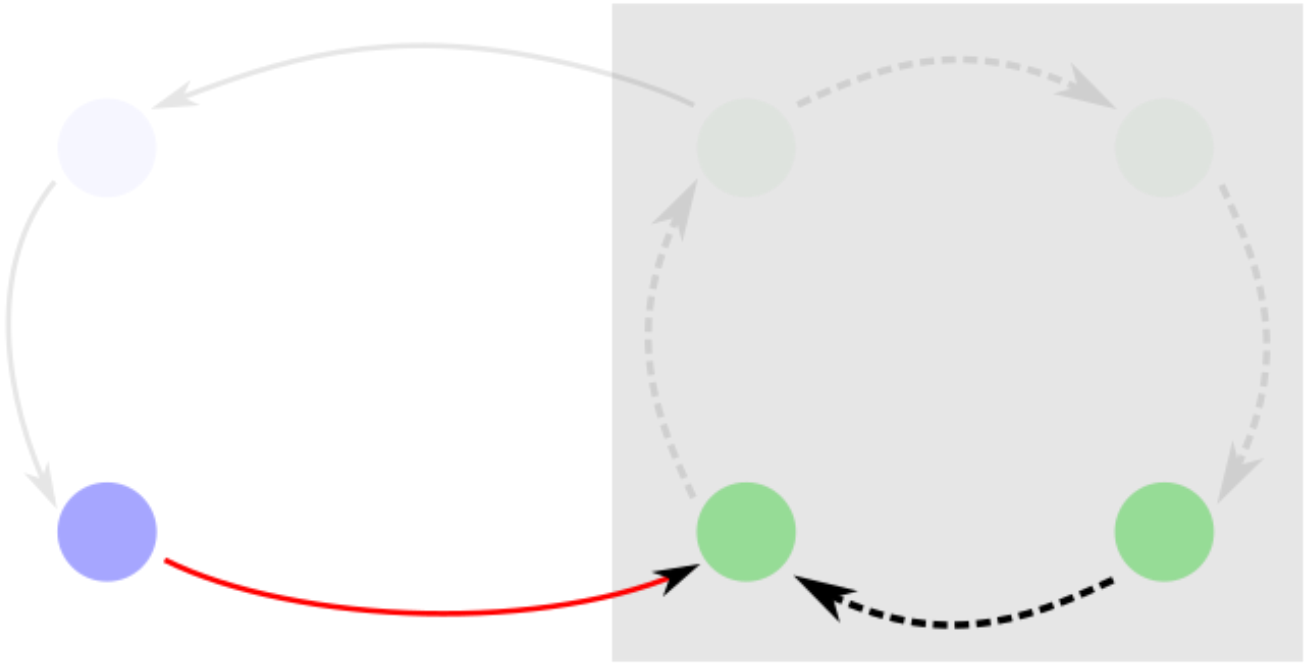

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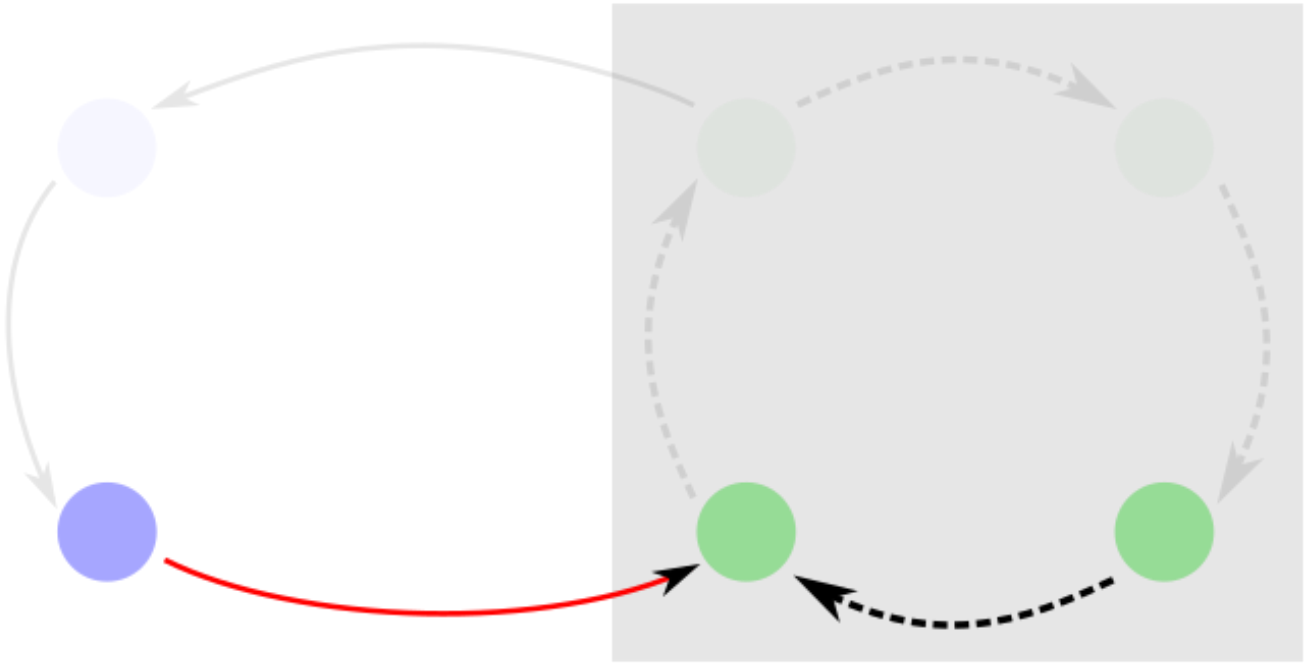

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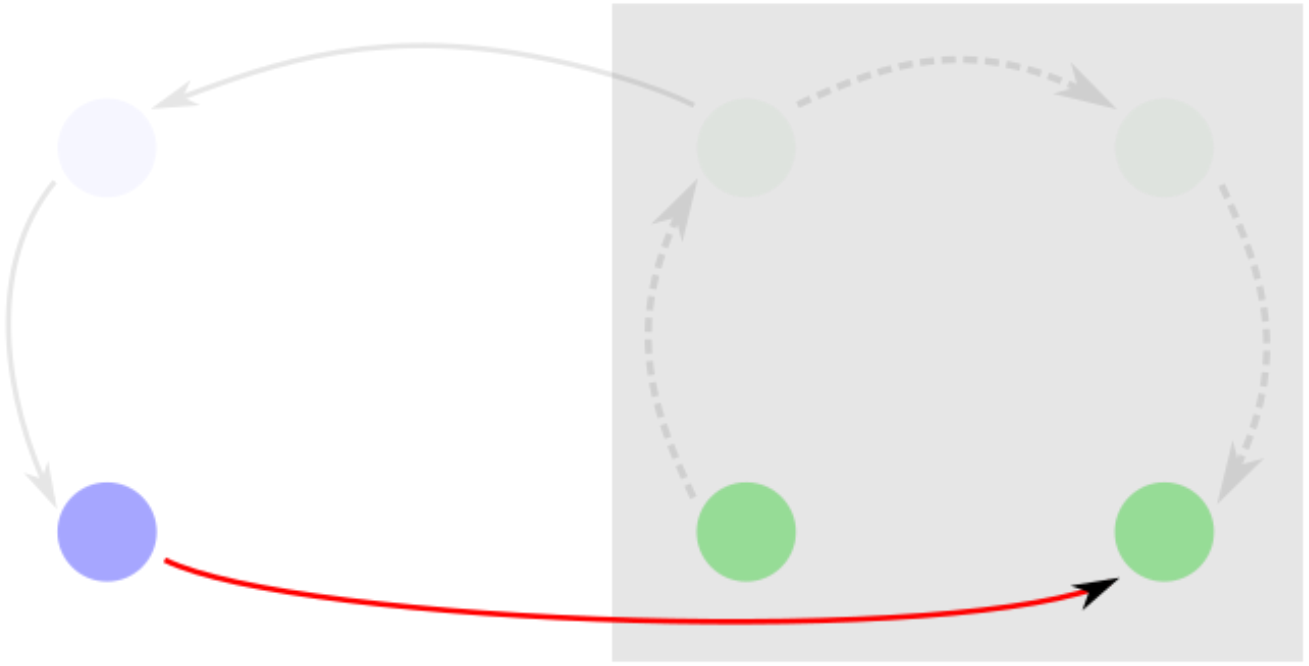

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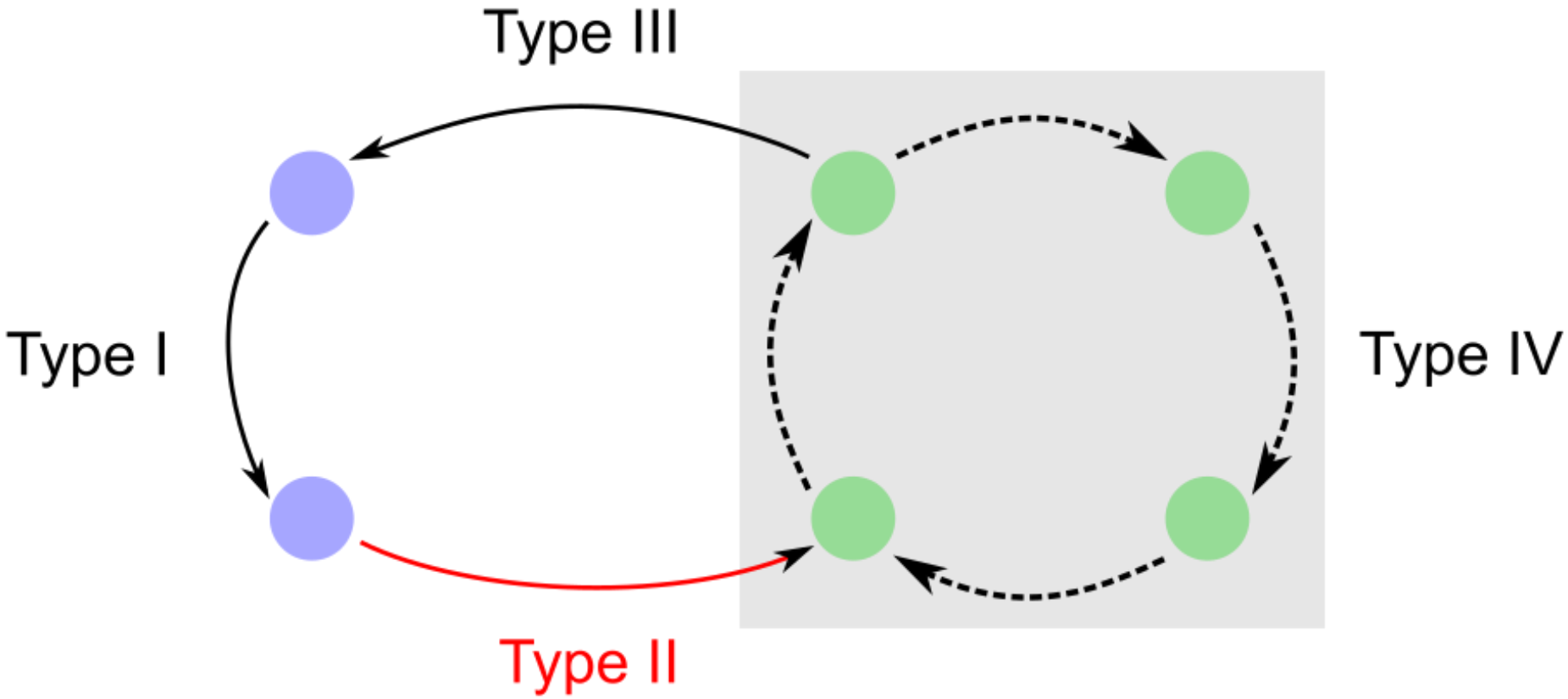

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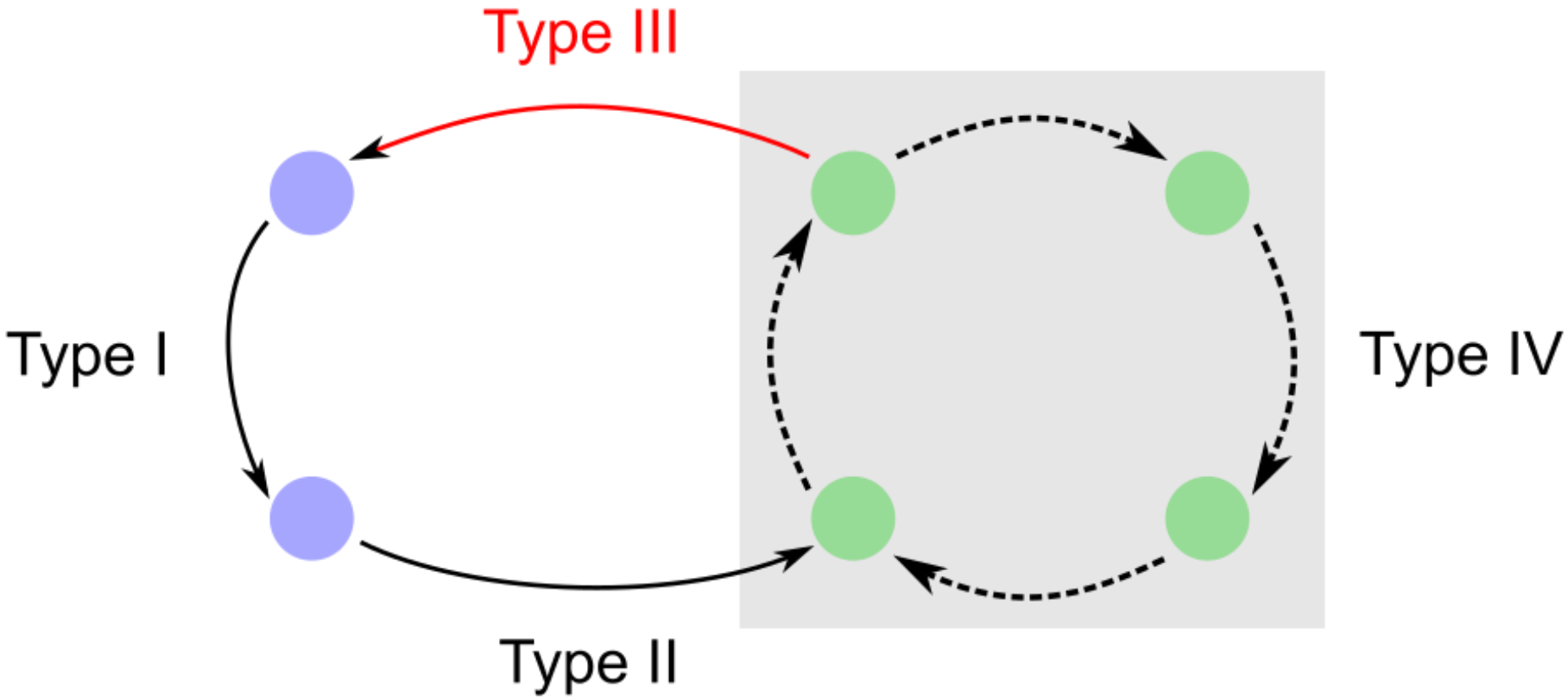

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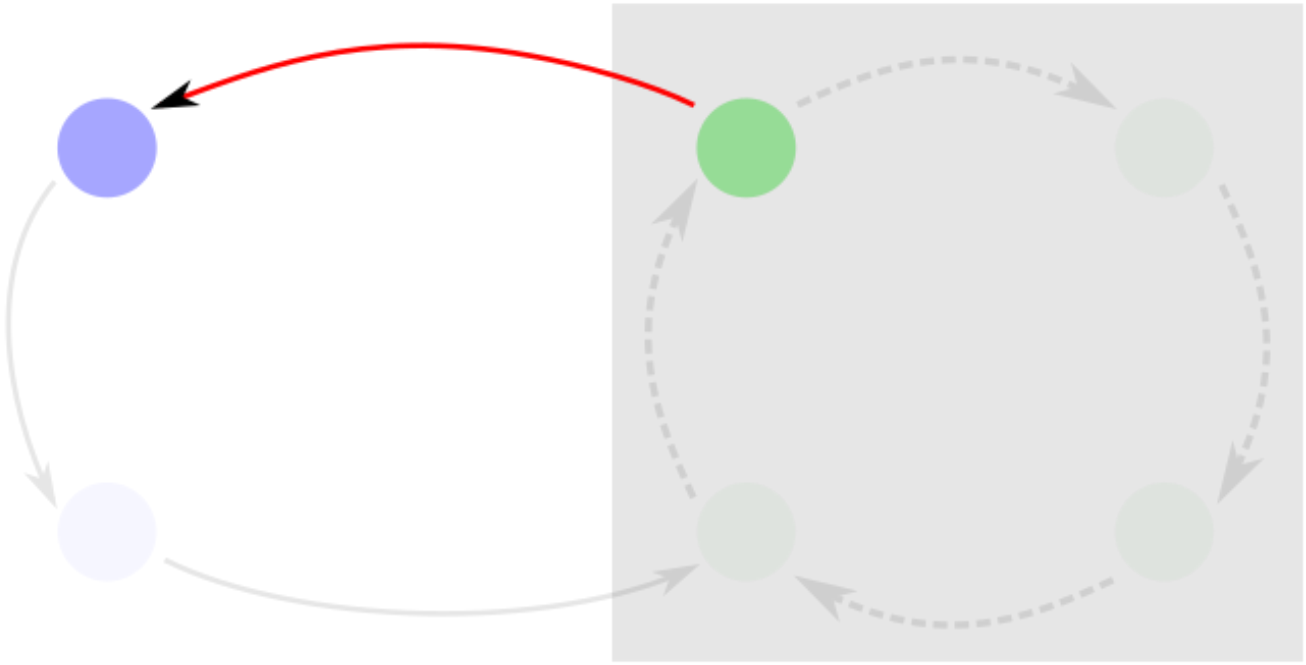

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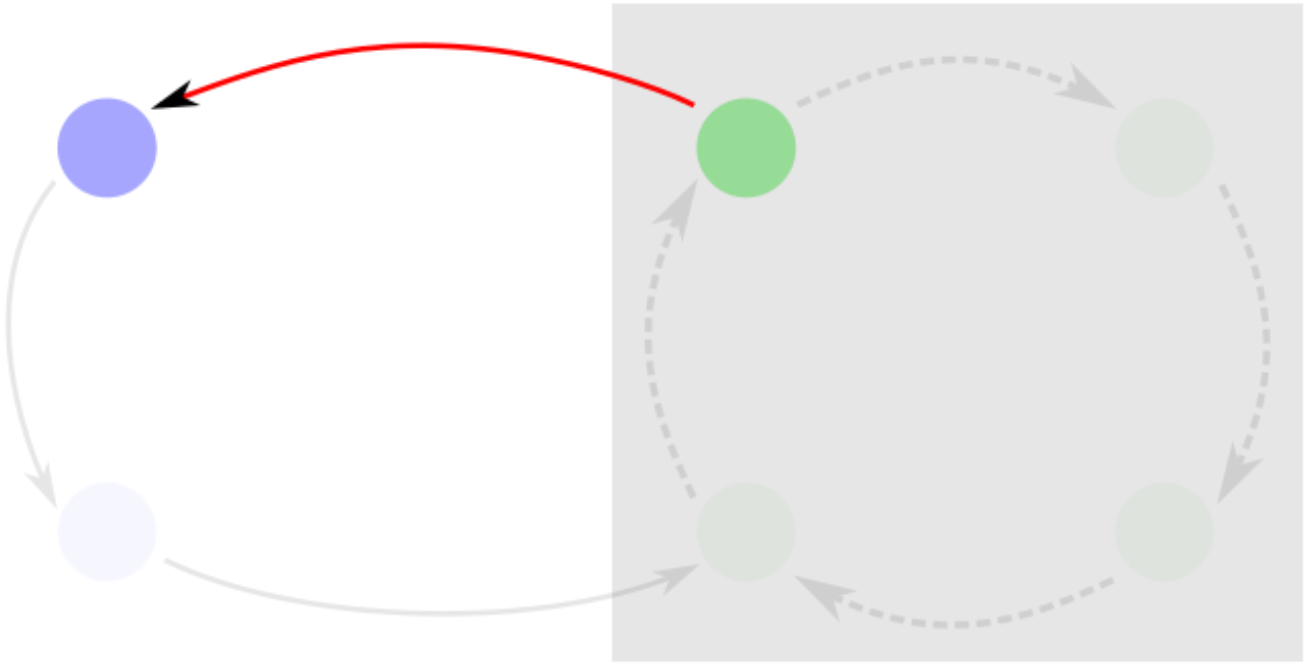

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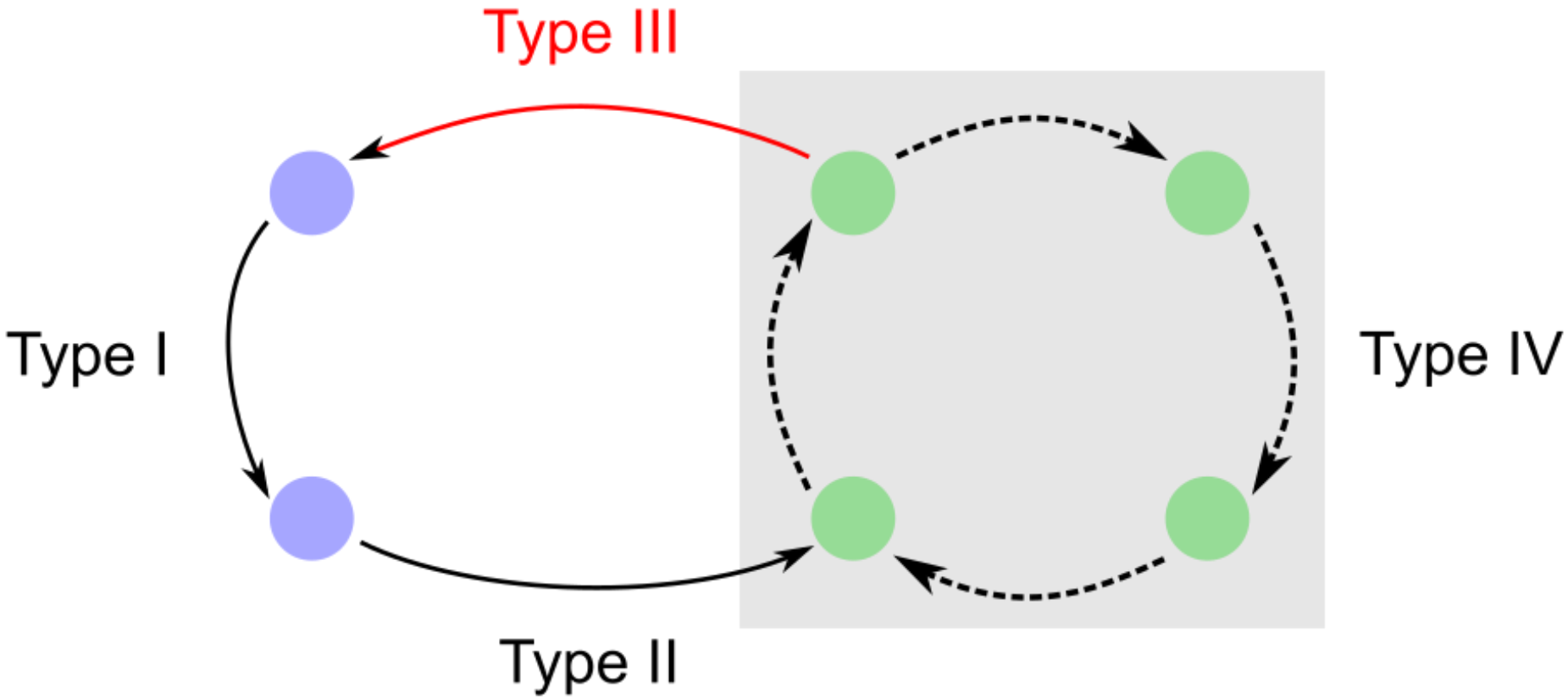

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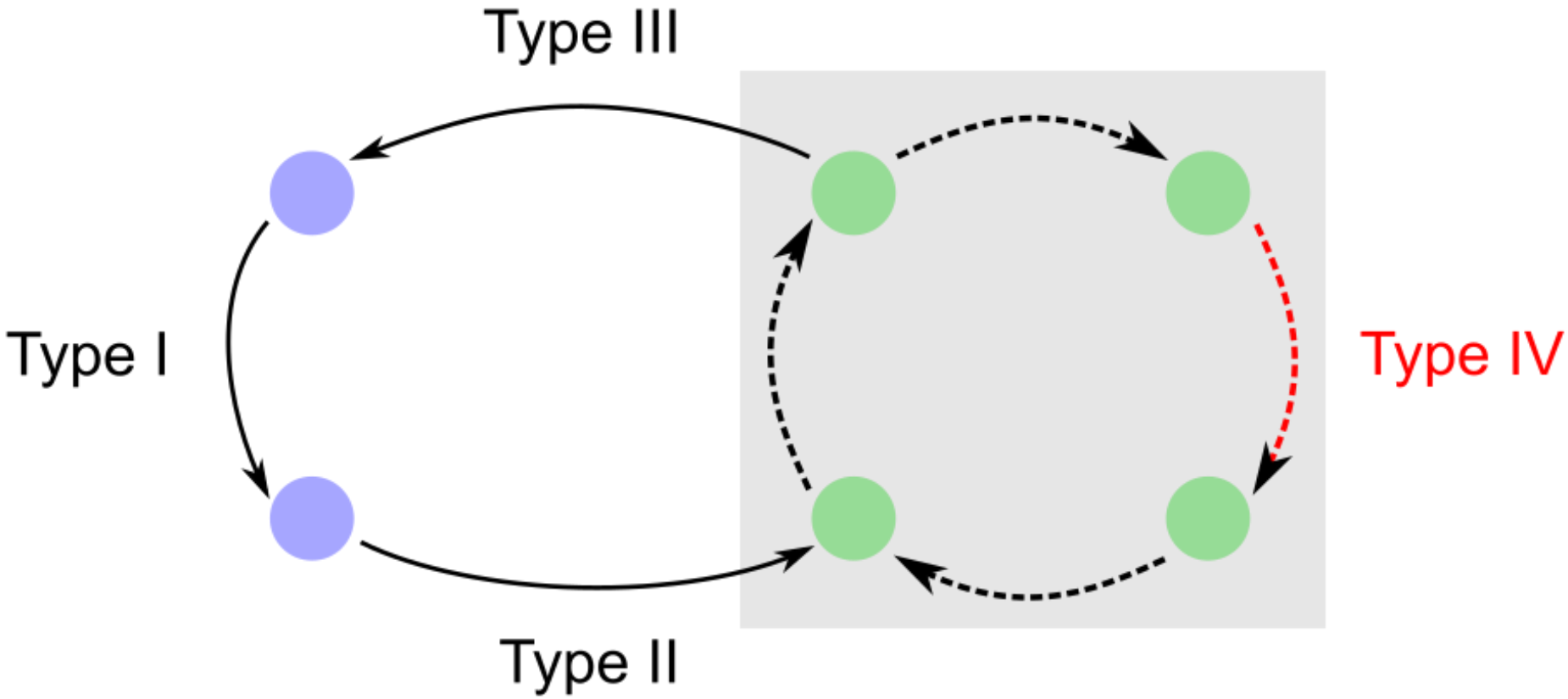

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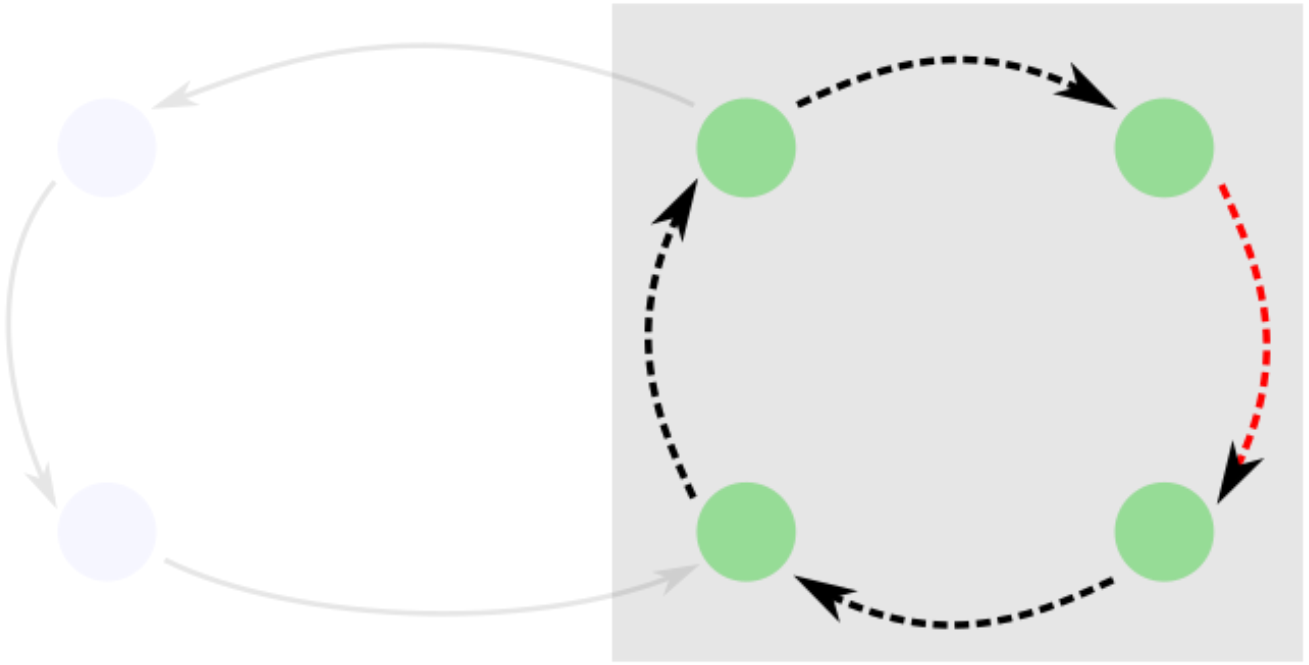

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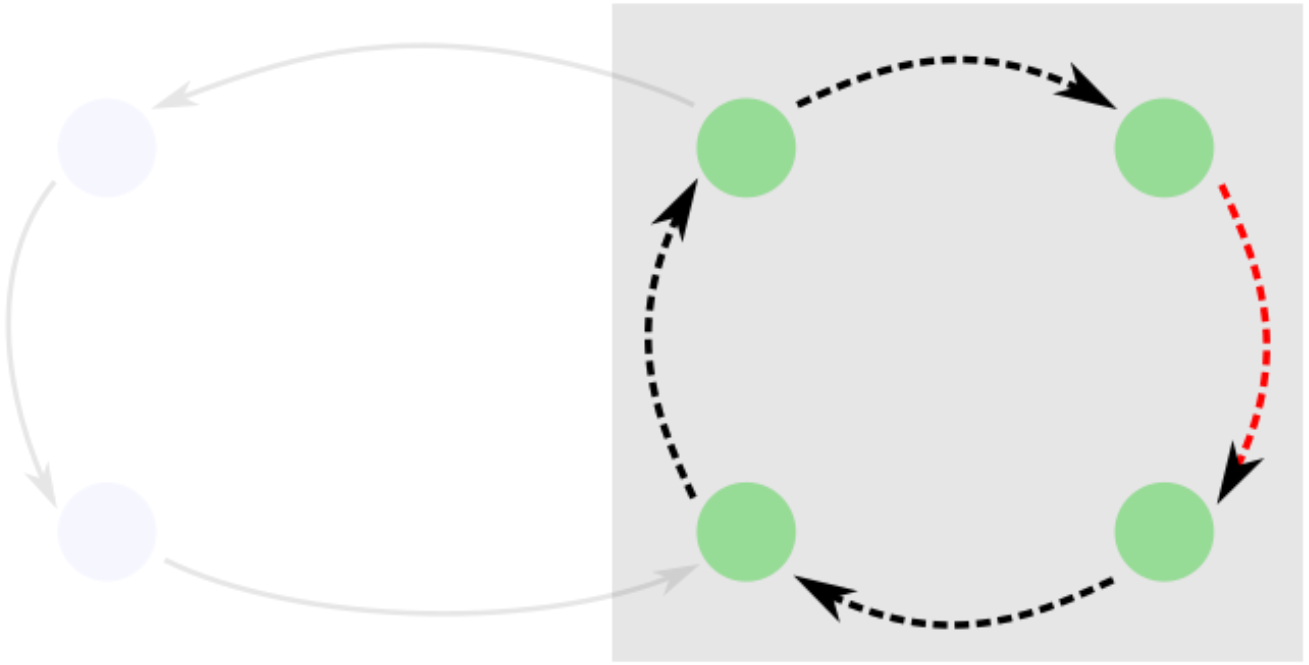

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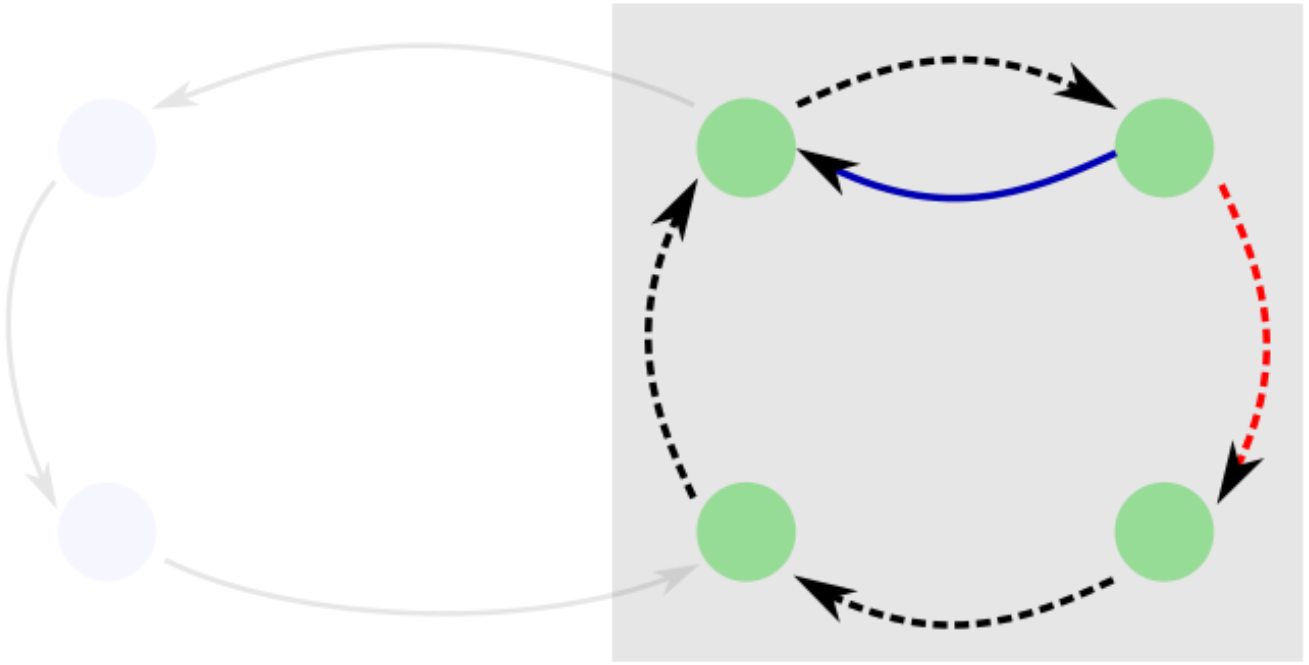

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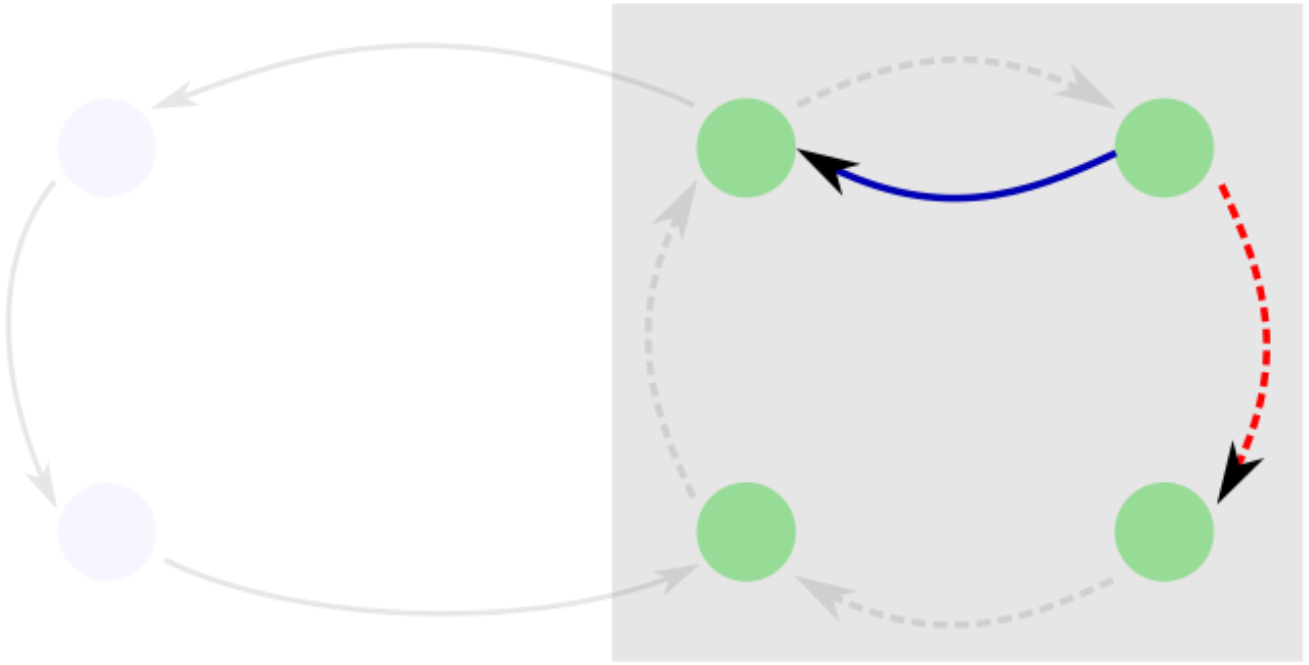

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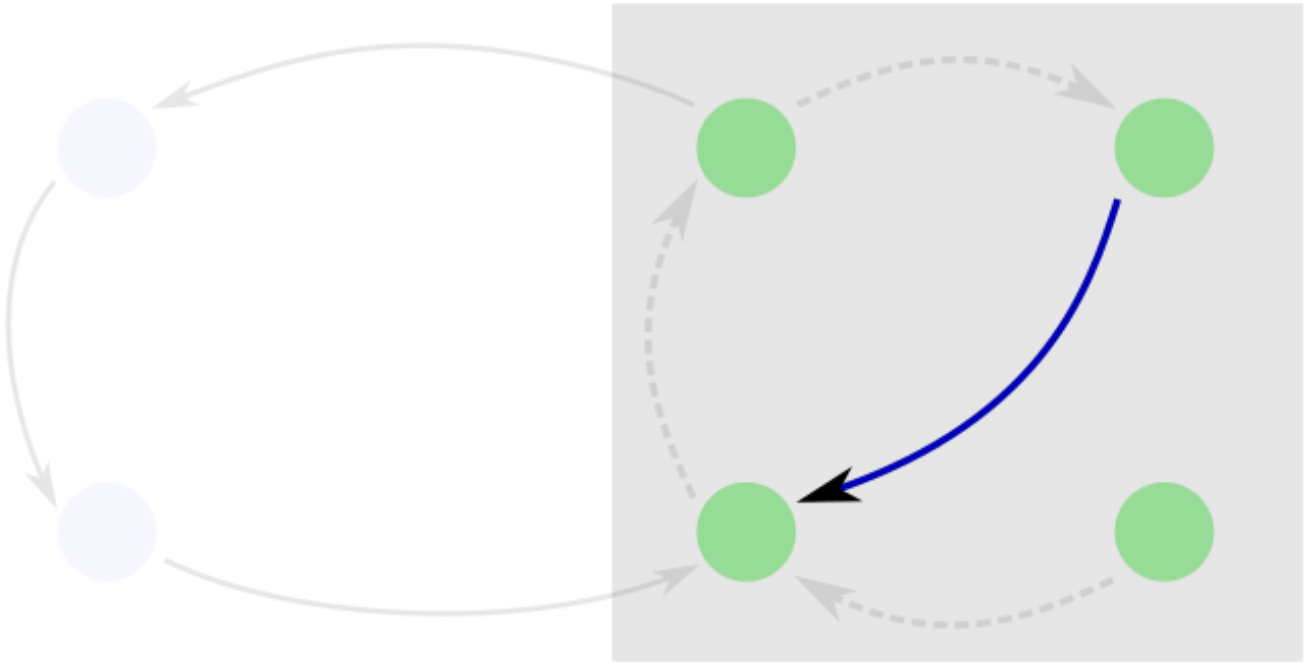

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Envy-cycle elimination algorithm

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Allocation A is EF1 if for every pair of agents i, k, there exists a good $j \in A_k$ such that $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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We will argue that each iteration of the algorithm "preserves" EF1.

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If the partial allocation at the beginning of an iteration is EF1, then the partial allocation at the end of that iteration is also EF1.

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Suppose good g is assigned to the source agent s. Then,

$$v_i(A_i) \geq v_i(A_s \cup \{g\} \setminus \{g\})$$

which means that EF1 is preserved.

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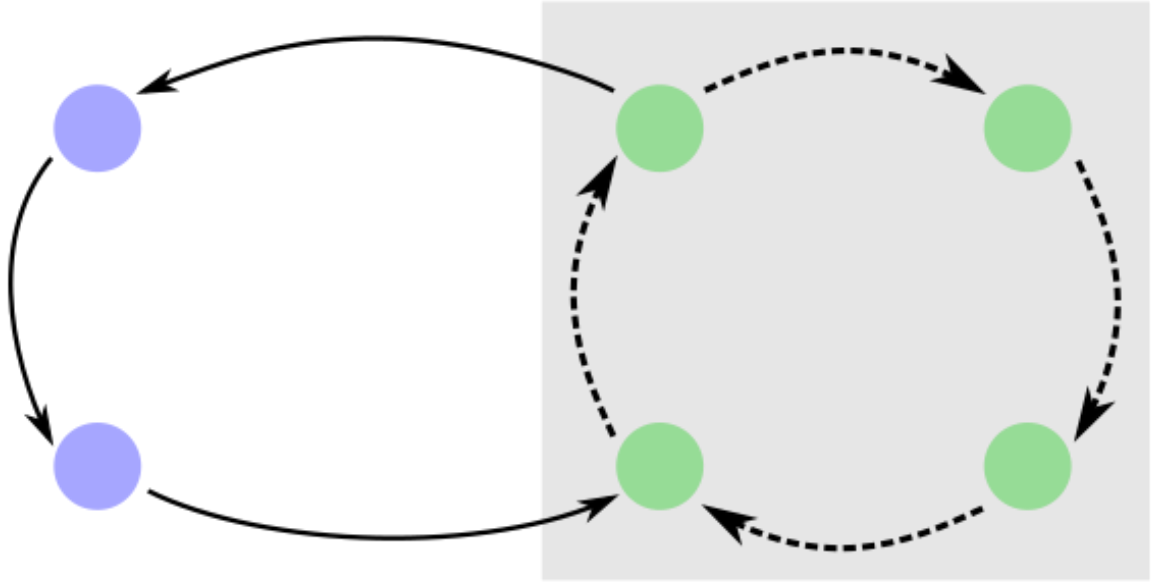

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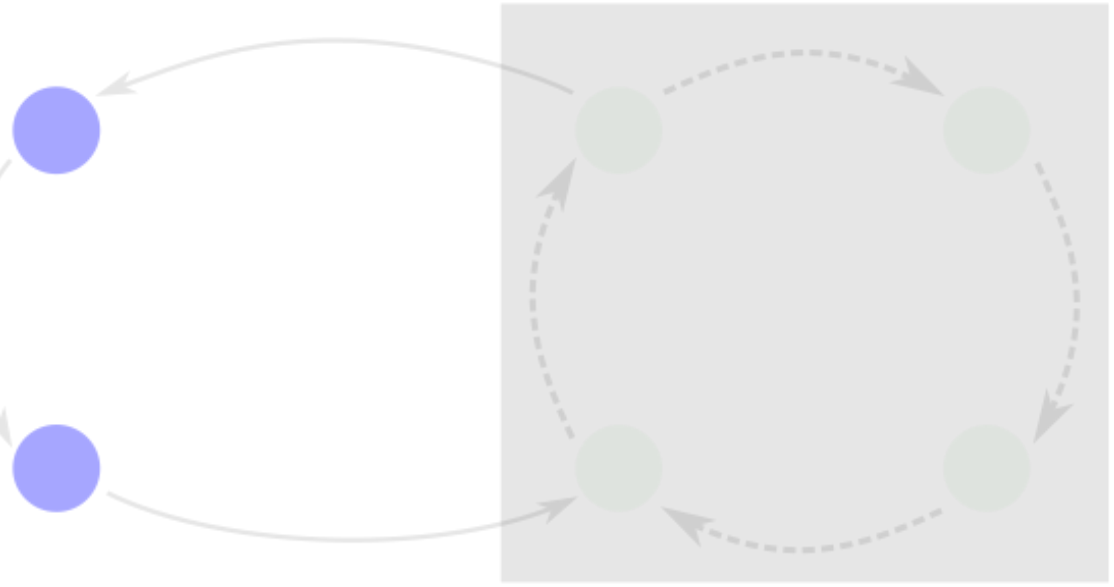
From their perspective, the bundles in the cycle are only shifted around. So, EF1 relations are the same as before.



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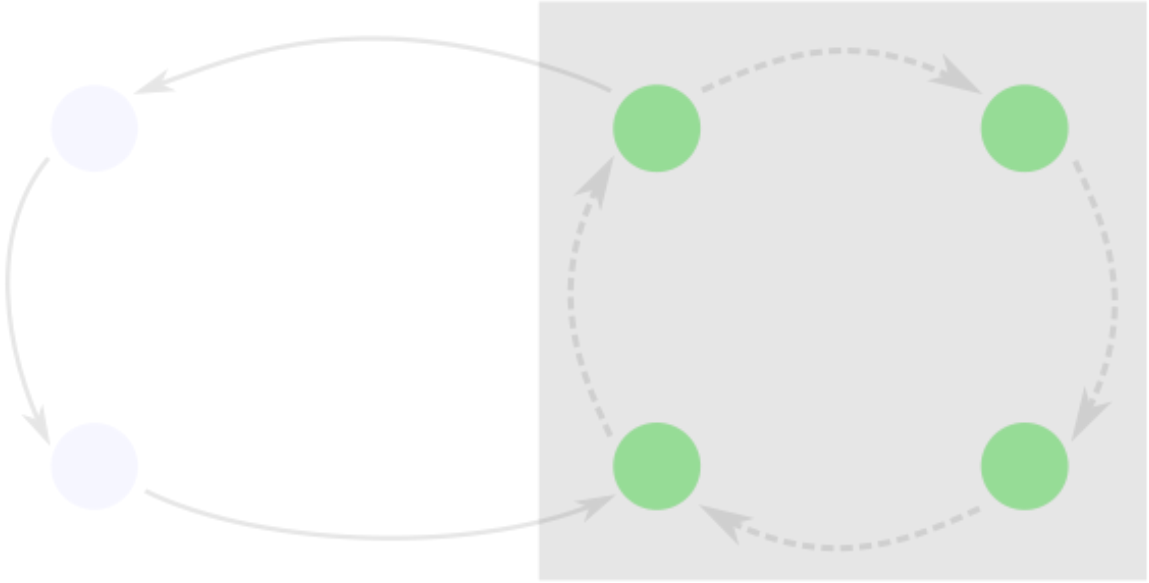

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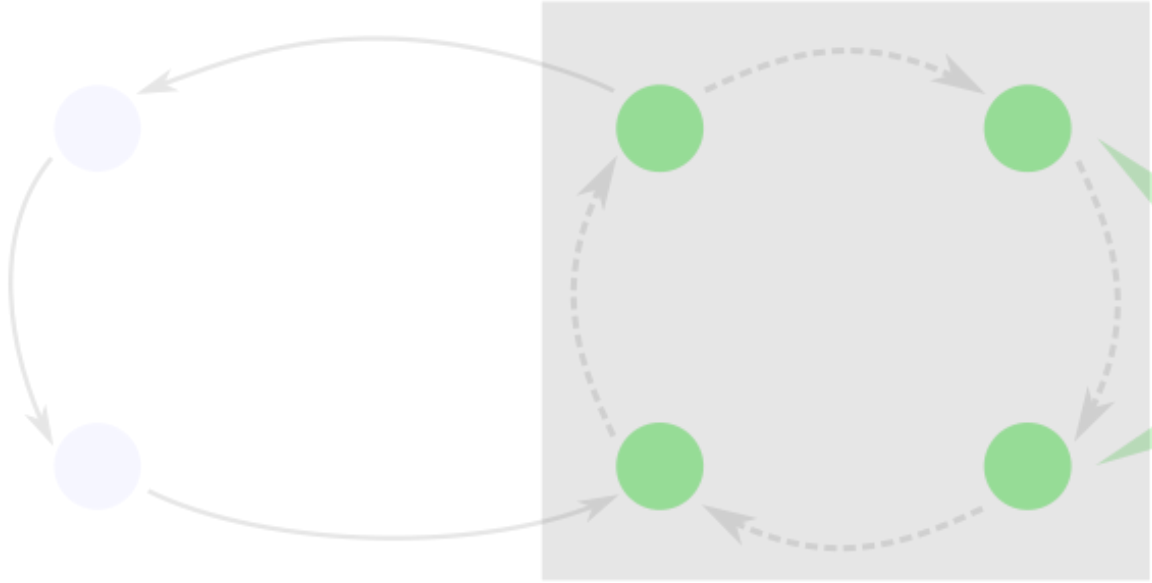

These agents are strictly better off, and any envied bundles are only shifted around. So, again, EF1 is maintained.

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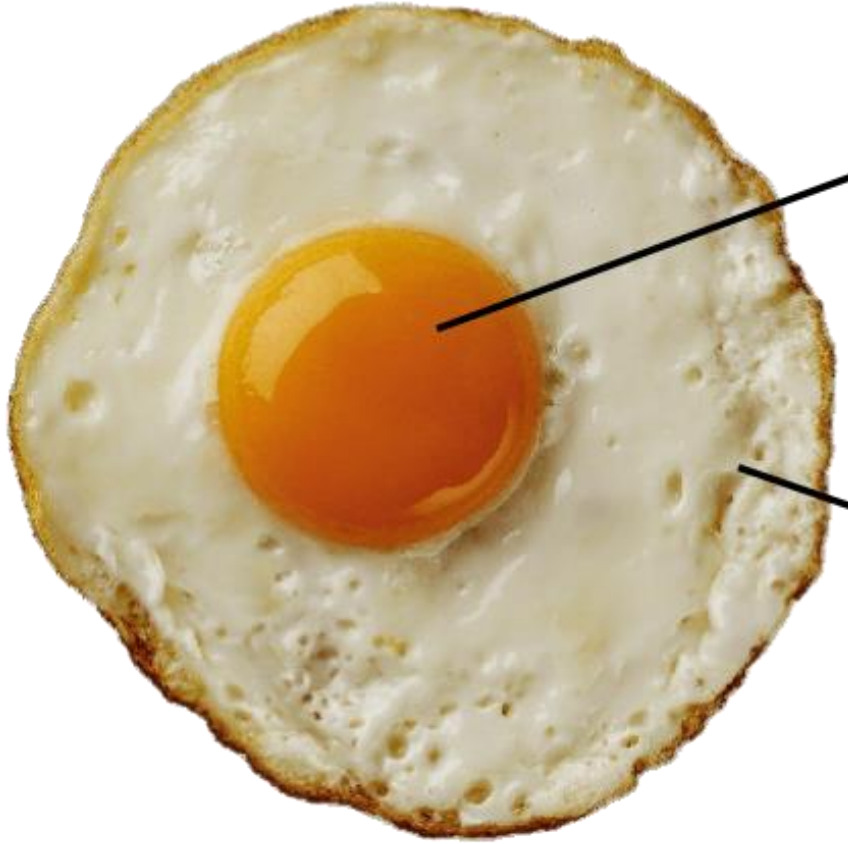

Additive: $v_i(S) = \sum_{j \in S} v_i(\{j\})$

Monotone: $S \subseteq T \Rightarrow v_i(S) \leq v_i(T)$

Next Time

Fairness and Efficiency

Quiz

Quiz

Prove or disprove:

For two agents, the round robin allocation is Pareto optimal.

An allocation A is Pareto optimal if there is no other allocation B such that:

- every agent is weakly better off under B, and
- some agent is strictly better off under B.

References

- Envy-cycle elimination algorithm

  Richard Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi
  "On Approximately Fair Allocations of Indivisible Goods"
  EC 2004, pg 125-131
  https://dl.acm.org/doi/10.1145/988772.988792